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Stable boundary spike clusters for the two-dimensional Gierer–Meinhardt system

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A B S T R A C T

We consider the Gierer–Meinhardt system with small inhibitor diffusivity and very small activator diffusivity in a bounded and smooth two-dimensional domain. For any given positive integer k we construct a spike cluster consisting of k boundary spikes which all approach the same nondegenerate local maximum point of the boundary curvature. We show that this spike cluster is linearly stable.

The main idea underpinning these stable spike clusters is the following: due to the small inhibitor diffusivity the interaction between spikes is repulsive and the spikes are attracted towards a nondegenerate local maximum point of the boundary curvature. Combining these two effects can lead to an equilibrium of spike positions within the cluster such that the cluster is linearly stable.

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R É S U M É

Nous considérons le système de Gierer–Meinhardt avec la petite diffusivité de l'inhibiteur et avec la très petite diffusivité de l'activateur dans un domaine de deux dimensions qui est limité et lisse. Pour toute donnée k entier positif nous construisons une grappe de spikes qui consiste en k spikes qui situé près de la frontière et qui approchent le point de maximum local non dégénéré du courbure de la frontière. Nous montrons que cette grappe de spikes est linéairement stable.

Le fondement de l'idée principale ces amas de spike stable est la suivante : en raison de la diffusivité de l'inhibiteur de la petite l'interaction entre pointes est répugnant et les pointes sont attirés vers un point de maximum local non dégénéré du courbure de la frontière. Combinant ces deux effets peut conduire à un équilibre des positions de spikes au sein du grappe tel que le grappe est linéairement stable.

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1. Introduction

Turing in his pioneering work in 1952 [16] proposed that a patterned distribution of two chemical substances, called the morphogens, could trigger the emergence of cell structures. He also gives the following explanation for the formation of the morphogenetic pattern: It is assumed that one of the morphogens, the activator, diffuses slowly and the other, the inhibitor, diffuses much faster. In the mathematical framework of a coupled system of reaction–diffusion equations with hugely different diffusion coefficients it is shown by linear stability analysis that the homogeneous state may be unstable. In particular, a small perturbation of spatially homogeneous initial data may evolve to a stable spatially complex pattern of the morphogens.

Since the work of Turing, many different reaction–diffusion system in biological modelling have been proposed and the occurrence of pattern formation has been investigated by studying what is now called Turing instability. One of the most popular models in biological pattern formation is the Gierer–Meinhardt system [7], see also [11]. In two dimensions in a special case after rescaling it can be stated as follows:

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \frac{u^2}{v}, u > 0 \text{ in } \Omega, \\ \tau v_t = D\Delta v - v + u^2, v > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

The unknown functions $u = u(x,t)$ and $v = v(x,t)$ represent the concentrations of the activator and inhibitor, respectively, at the point $x \in \Omega \subset \mathbb{R}^2$ and at a time $t > 0$. Here Δ is the Laplace operator in $\mathbb{R}^2$, Ω is a smooth bounded domain in $\mathbb{R}^2$, $\nu = \nu(x)$ is the outer unit normal at $x \in \partial\Omega$.

Throughout this paper, we assume that

$$0 < \varepsilon \ll 1, \; 0 < D \ll 1, \tag{1.2}$$

$\tau \geq 0$ is a fixed constant independent of ε, D and x. Further, the diffusivities ε and D do not depend on x but they are both small constants. In this paper, we further assume that

$$e^{-\frac{1}{\sqrt{D}}} \ll \varepsilon \ll \sqrt{D}. \tag{1.3}$$

This means that ϵ is much smaller than D. On the other hand, ϵ cannot be exponentially small compared to $\sqrt{D}$.

In this paper, we study the Gierer–Meinhardt system in a bounded and smooth two-dimensional domain. We prove the existence and stability of a cluster consisting of k boundary spikes near a nondegenerate local maximum point P^0 of the boundary curvature $h(P)$.

A spike cluster is the combination of several spikes which all approach the same point in the singular limit. The main idea underpinning these stable spike clusters is the following: due to the small inhibitor diffusivity the interaction between spikes is repulsive and the spikes are attracted towards a nondegenerate local maximum point of the boundary curvature. Combining these two effects can lead to an equilibrium of spike positions within the cluster such that the cluster is linearly stable.

Highlights of the Gierer–Meinhardt system in this setting include the following: it contains three different length scales: $O(1)$ scale of boundary curvature, $O(\sqrt{D})$ scale of inhibitor diffusivity and $O(\epsilon)$ scale of activator diffusivity; it is biologically relevant since it can model a hierarchical process (pattern formation of small-scale structures induced by the boundary of a pre-existing large-scale domain; the expressions for spike positions and eigenvalues can be made explicit and often have a particularly simple form.

The spike cluster solutions considered in this paper show multiple scales which appear in a robust and stable manner. A real-world biological example incorporating multiple scales similar to those in spike clusters is the pattern formation of the head (more precisely, hypostome), tentacles, and foot in *hydra*. Meinhardt's model [12] correctly predicts the following experimental observation: with tentacle-specific antibodies, Bode et al. [2] have shown that tentacle formation is a two-stage process: (i) after head removal tentacle activation first reappears at the very tip of the gastric column; (ii) then this activation becomes shifted away from the tip to a new location, where the tentacles eventually appear. This tentacle pattern incorporates multiple lengthscales such as the diameter of the gastric column, the distance between tentacles, and the diameter of tentacles. Both the dynamical process and the final pattern of tentacle formation resemble the behaviour of the Gierer–Meinhardt system considered in this paper.

Let us now summarize the analytical approach employed in our paper. The existence proof is based on Liapunov–Schmidt reduction. The stability of the cluster is shown by first separating the eigenvalues into two cases: large eigenvalues which tend to a nonzero limit and small eigenvalues which tend to zero in the limit $D \to 0$ and $\frac{\epsilon}{\sqrt{D}} \to 0$. Large eigenvalues are then explored by deriving suitable nonlocal eigenvalue problems whose stability follows from [18]. Small eigenvalues are calculated explicitly by an using asymptotic analysis with rigorous error estimates for which the curvature of the boundary plays the key role similar to the analysis in [19]. However, in this paper, due to properties of the spike cluster, the small eigenvalues are of two different orders, whereas in [19] all small eigenvalues have the same order.

Before we state our main results, let us mention some previous ones concerning various regimes for the asymptotic behaviour of D.

The repulsive nature of spikes has been shown in [6]. The existence and stability of a spike cluster made up of two boundary spikes has been established in [5].

For the strong coupling case, i.e. $D \sim 1$, the second and third authors constructed single-interior spike solutions [22]. In [24], they continued the study, and proved the existence of solutions with k interior spikes.

Moreover, it is shown that this solution is linearly stable for $\tau = 0$.

For the weak coupling case $D \to \infty$, in [23] the second and third authors proved the existence of multiple interior spike solutions.

Further, they showed that there are stability thresholds

$$D_1(\varepsilon) > D_2(\varepsilon) > \cdots > D_k(\varepsilon) > \cdots$$

such that if $\lim_{\varepsilon\to 0} \frac{D_k(\varepsilon)}{D} > 1$, the k-peak solution is stable and if $\lim_{\varepsilon\to 0} \frac{D_k(\varepsilon)}{D} < 1$, the k-peak solution is unstable. Multiple spikes for the Gierer–Meinhardt system in a one-dimensional interval have been studied in [15,9,25] and on the real line in [4].

In [19] the existence, uniqueness and spectral properties of a boundary spike solution have been studied for the shadow Gierer–Meinhardt system (i.e. after formally taking the limit $D \to \infty$).

In [26] the existence and stability of N-peaked steady states for the Gierer–Meinhardt system with precursor inhomogeneity has been explored. The spikes in the patterns can vary in amplitude. In particular, the results imply that a precursor inhomogeneity can induce instability. Single-spike solutions for the Gierer–Meinhardt system with precursor including spike dynamics have been studied in [17].

Previous results on stable spike clusters include a stable spike cluster for a consumer chain model [27] and a stable spike cluster for the one-dimensional Gierer–Meinhardt system with precursor inhomogeneity [29].

Polygonal spike patterns for the Gierer–Meinhardt system in the two-dimensional plane have been derived in [3]. Polygonal stable spike clusters have been considered in [30] in the interior of a bounded two-dimensional domain near a local minimum of a precursor inhomogeneity. Work on polygonal stable spike clusters for the Gierer–Meinhardt system on a two-dimensional Riemannian manifold near a local maximum of the Gaussian curvature is in progress [1].

For more background, modelling, analysis and computation on the Gierer–Meinhardt system, we refer to [28] and the references therein.

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2. Main results: existence and stability

Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth two-dimensional domain. Let w be the unique solution in $H^1(\mathbb{R}^2)$ of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \\ w(0) = \max_{y\in\mathbb{R}^2} w(y), \ \ w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.1}$$

For the existence and uniqueness of (2.1), we refer to [10] and [13]. We also recall that w is radially symmetric and

$$w(y) \sim |y|^{-\frac{1}{2}} e^{-|y|} \quad \text{ as } |y| \to \infty$$

and

$$w'(y) = -(1 + o(1))w(y) \quad \text{ as } |y| \to \infty,$$

where w' is the radial derivative of w, i.e. $w' = w_r(r)$.

Theorem 2.1. *Let k be a positive integer, and P^0 be a nondegenerate local maximum point of the curvature $h(P)$ of the boundary $\partial\Omega$.*

Then for $0 < e^{-\frac{1}{\sqrt{D}}} \ll \varepsilon \ll \sqrt{D} \ll 1$, the Gierer–Meinhardt system (1.1) has a k-boundary spike cluster steady-state solution (u_ϵ, v_ϵ) which concentrates near P^0. In particular, it satisfies

$$u_\epsilon \sim \frac{D\xi_\sigma}{\epsilon^2} \sum_{i=1}^{k} w(\frac{x - P_{i,\epsilon}}{\epsilon}),$$

where $P_{i,\epsilon} \to P^0$ as $\epsilon \to 0$ for $i = 1, 2, \ldots, k$.

Further, we have

$$\xi_\sigma \sim \frac{1}{\log \frac{\sqrt{D}}{\epsilon}}$$

and

$$|P_i - P_{i-1}| \sim \sqrt{D} \log \frac{\xi_\sigma}{\epsilon D}, \quad i = 2, \ldots, k.$$

Remark 2.1. The spike cluster is established by a balance between repelling spikes and attracting boundary point of local maximum curvature.

Theorem 2.2. *The k-boundary spike cluster solution given in Theorem 2.1 is linearly stable if τ is small enough.*

Remark 2.2. There are eigenvalues of two different orders: $n-1$ eigenvalue related to repelling of neighbouring spikes are of order $\epsilon^3 \log \frac{\xi_\sigma}{\epsilon D}$, and one eigenvalue stemming from the curvature of the boundary (corresponding to synchronous motion of all spikes) is of order ϵ^3.

We confirm and illustrate (see Figs. 1–4) the main results by a few numerical computations which have been performed using the Software COMSOL. The patterns shown have been obtained as long-term limits

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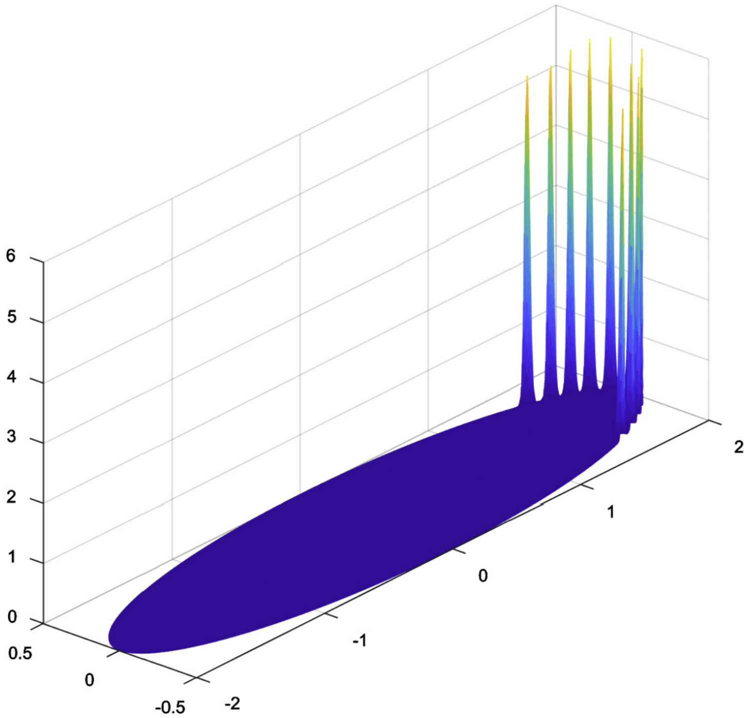

Fig. 1. Clustered spiky steady state of (1.1) for $\epsilon^2 = 0.00004$, $D = 0.001$. Shown is a boundary spike cluster consisting of 11 spikes. The activator A is displayed in a three-dimensional surface plot.

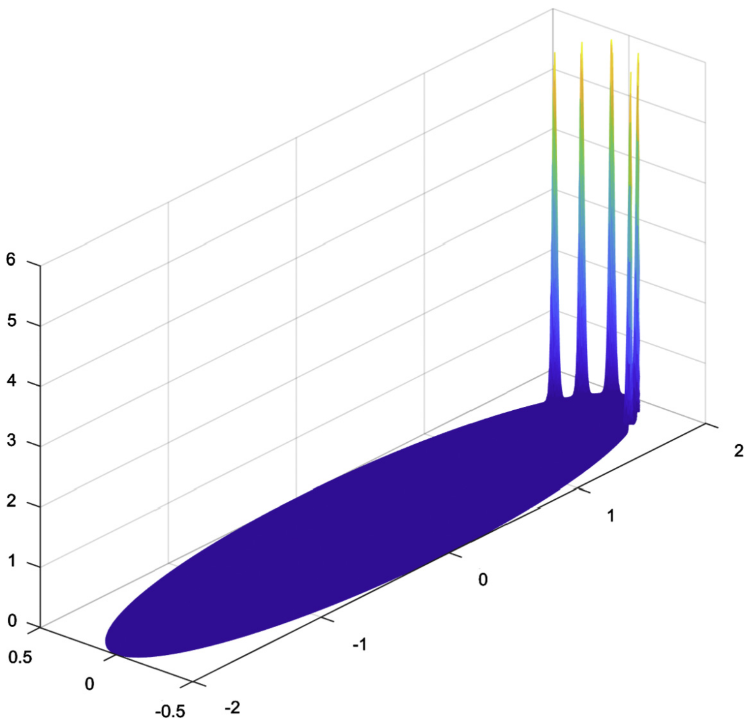

Fig. 2. Clustered spiky steady state of (1.1) for $\epsilon^2 = 0.00004$, $D = 0.001$. Shown is a boundary spike cluster consisting of 6 spikes.

of the time-dependent Gierer–Meinhardt system. Initial conditions are chosen as follows: the activator possesses a sharp peak near the maximum point of mean curvature combined with small-scale oscillations at the boundary, e.g. given by

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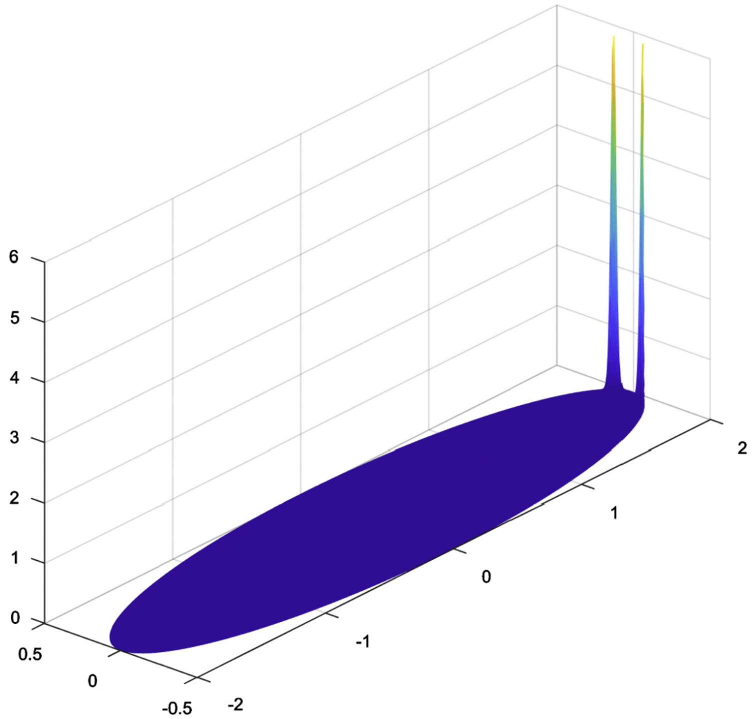

Fig. 3. Clustered spiky steady state of (1.1) for $\epsilon^2 = 0.00004$, $D = 0.001$. Shown is a boundary spike cluster consisting of 2 spikes.

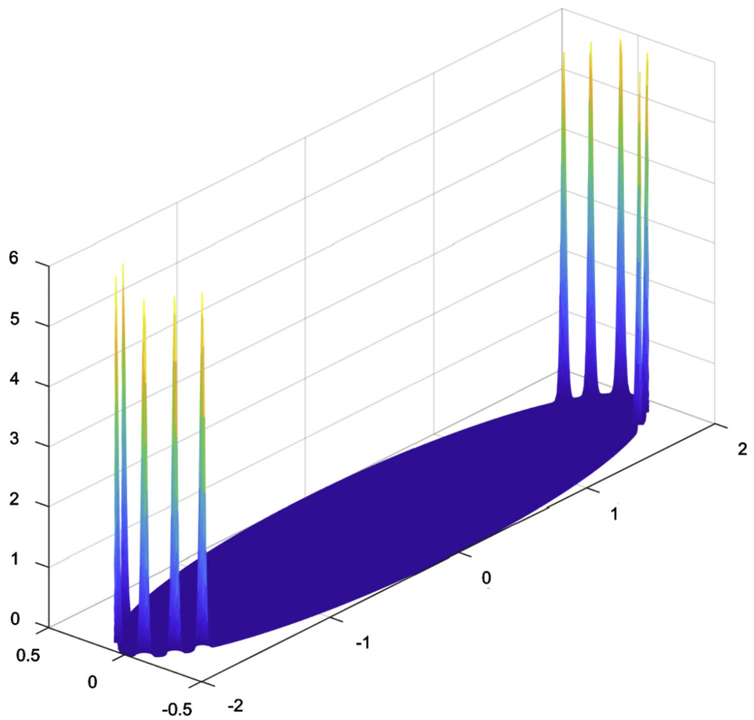

Fig. 4. Clustered spiky steady states of (1.1) for $\epsilon^2 = 0.00004$, $D = 0.001$. Shown is a steady state of two boundary spike clusters consisting each of 6 spikes.

$$u(x,0) = 3e^{-100000(1-0.25x_1^2-4x_2^2)}e^{-100\phi}(2 - \cos(1000\phi))$$

close to the point $x_1 = 2, x_2 = 0$, where $\phi = \arctan(x_2/x_1)$. For the inhibitor we have simply taken $v(x,0) = 0$.

This paper is organised as follows. In sections 3–5 we show existence of the spike cluster steady state by using Liapunov–Schmidt reduction. In section 3 we introduce an approximation to the spike cluster steady state. In section 4 we use the Liapunov–Schmidt method to reduce the problem to finite dimensions. In section 5 we solve this reduced problem. In sections 6–7 we study the stability of this spike cluster steady state. In section 6 we consider large eigenvalues. Finally, in section 7 we study small eigenvalues. In two appendices we show some technical results: in appendix A (section 8) we prove Proposition 4.1 and in appendix B (section 9) we compute the small eigenvalues.

3. Introduction of the approximate solutions

In this paper, we consider the following problem:

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^2}{v} = 0 \text{ in } \Omega, \\ D\Delta v - v + u^2 = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and smooth two-dimensional domain.

Let P^0 be a nondegenerate maximum point of the boundary curvature $h(P)$ on the boundary of Ω. For $P \in \partial\Omega$,

$$\nabla_{\tau(P)} := \frac{\partial}{\partial \tau(P)},$$

where $\frac{\partial}{\partial \tau(P)}$ denotes the tangential derivative with respect to P at $P \in \partial\Omega$. We will sometimes drop the variable P if this can be done without causing confusion.

In this section, we construct an approximation to a spike cluster solution to (3.1) which concentrates at P^0.

The approximate cluster consists of spikes $\sigma^{-2}\xi_{\sigma,i}w(\frac{x-P_i}{\varepsilon})$ which are centred at the points P_i for $i = 1, \cdots, k$, where $\sigma = \frac{\varepsilon}{\sqrt{D}}$ and the amplitude $\xi_{\sigma,i}$ satisfies

$$\xi_{\sigma,i} \sim \frac{1}{\frac{1}{\pi}\log\frac{1}{\sigma}\int_{\mathbb{R}^2_+} w^2\,dx}$$

(see (3.38)).

Let $P_1, \cdots, P_k$ be k points distributed along the boundary $\partial\Omega$ such that we have for $i = 2, \cdots, k$

$$\left| \frac{P_i - P_{i-1}}{\sqrt{D}} - \log\frac{\xi_\sigma}{\varepsilon D} + \frac{3}{2}\log\log\frac{\xi_\sigma}{\varepsilon D} + \log\Big(-\frac{\frac{\partial^2}{\partial\tau^2}h(P^0)\nu_1}{2\nu_2}\Big) + \log[(i-1)(k+1-i)] \right| \le \eta \tag{3.2}$$

and

$$\left| \frac{1}{k}\sum_{i=1}^{k} P_i - P^0 \right| \le \eta\sqrt{D}\log\frac{\xi_\sigma}{\varepsilon D}, \tag{3.3}$$

where ν_2 is given in (5.10) below. Further, $\eta > 0$ is a small constant independent of ε and D. The reason for assuming (3.2) and (3.3) will become clear in Section 5 when we solve the reduced problem.

Remark 3.1. By (3.2), the distance of neighbouring spikes satisfies

$$|P_i - P_{i-1}| \sim C\sqrt{D} \log \frac{\xi_\sigma}{\varepsilon D}.$$

Since we want to construct multiple boundary spikes which collapse at one point, we require that assumptions (1.2) and (1.3) hold.

After re-scaling,

$$\hat{u}(z) = \sigma^2 u(\varepsilon z), \quad \hat{v}(z) = \sigma^2 v(\varepsilon z),$$

if we drop the hat and still denote solutions by (u, v), equation (3.1) is equivalent to

$$\begin{cases} \Delta u - u + \frac{u^2}{v} = 0 \text{ in } \Omega_\varepsilon, \\ \Delta v - \sigma^2 v + u^2 = 0 \text{ in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{3.4}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$.

From now on we will deal with (3.4). Before introducing an approximation to the spike solutions, we first define some notation.

Fixing $\mathbf{P}^\varepsilon = (P_1, \cdots, P_k)$ such that (3.2) and (3.3) hold, we set

$$\Lambda_k = \{\mathbf{p}^\varepsilon = \varepsilon^{-1}\mathbf{P}^\varepsilon \,:\, P_1, \cdots, P_k \text{ such that (3.2) and (3.3) hold }\}. \tag{3.5}$$

We are looking for multiple spike solutions to (3.4) of the form

$$\begin{cases} u(z) \sim \sum_{i=1}^k \xi_{\sigma,i} P w_{p_i}(z - p_i), \\ v(p_i) \sim \xi_{\sigma,i}, \end{cases} \tag{3.6}$$

where $Pw_{p_i}(z - p_i)$ is defined to be the unique solution of

$$\Delta u - u + w(\cdot - p_i)^2 = 0 \text{ in } \Omega_{\varepsilon,p_i}, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\varepsilon,p_i}. \tag{3.7}$$

Here $\Omega_{\varepsilon,p_i} = \{z \,:\, z + p_i \in \Omega_\varepsilon\}$, the function w has been defined in (2.1) and $\xi_{\sigma,i}, \ i = 1, \cdots, k$ are the heights of the spikes, which will be determined in (3.38).

3.1. The analysis of the projection $Pw_q(z - q)$

Before calculating the heights of the spikes, we need some preliminaries of the projection $Pw_{p_i}(z - p_i)$ defined in (3.7) which are rather standard by now. Some of these results have been derived in [20,21].

Let $P \in \partial\Omega$. We define a diffeomorphism straightening the boundary. We may assume that the inward normal to $\partial\Omega$ at P is pointing in the direction of the positive x_2 axis. Denote $B'(R) = \{x \in \mathbb{R}^2 | |x_1| \le R\}$. Then since $\partial\Omega$ is smooth, we can find a constant R such that $\partial\Omega$ can be represented by the graph of a smooth function $\rho_P : B'(R) \to \mathbb{R}$, where $\rho_P(0) = 0$, and $\rho'_P(0) = 0$. From now on, we omit the use of P in ρ_P and write ρ if this can be done without causing confusion. So near P, $\partial\Omega$ can be represented by $(x_1, \rho(x_1))$. The curvature of $\partial\Omega$ at p is $h(P) = \rho''(0)$. Let $\Omega_1 = \Omega \cap B(P, R) = \{(x_1, x_2) \in B(P, R) | x_2 - P_2 > \rho(x_1 - P_1)\}$, where $B(P, R) = \{x \in \mathbb{R}^2 | |x - P| < R\}$.

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After rescaling, it follows that near $p = \frac{P}{\varepsilon}$, the boundary $\partial\Omega_\varepsilon$ can be represented by $(z_1 - \mathbf{p}_1, \varepsilon^{-1}\rho(\varepsilon(z_1 - \mathbf{p}_1)))$, where $(z_1, z_2) = \varepsilon^{-1}(x_1, x_2)$ and $p = (\mathbf{p}_1, \mathbf{p}_2)$. By Taylor expansion, we have

$$\varepsilon^{-1}\rho(\varepsilon(z_1 - \mathbf{p}_1)) = \frac{1}{2}\rho''(0)\varepsilon(z_1 - \mathbf{p}_1)^2 + \frac{1}{6}\rho^{(3)}(0)\varepsilon^2(z_1 - \mathbf{p}_1)^3 + O(\varepsilon^3(z_1 - \mathbf{p}_1)^4). \tag{3.8}$$

Let $h_p(z) = w(z - p) - Pw_p(z - p)$. Then h_p satisfies

$$\begin{cases} \Delta h_p(z) - h_p(z) = 0, \text{ in } \Omega_\varepsilon, \\ \frac{\partial h_p}{\partial \nu} = \frac{\partial}{\partial \nu} w(z - p) \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{3.9}$$

For $z \in \Omega_{1\varepsilon} = \frac{1}{\varepsilon}\Omega_1$ we set

$$\begin{cases} y_1 = z_1 - \mathbf{p}_1, \\ y_2 = z_2 - \mathbf{p}_2 - \varepsilon^{-1}\rho(\varepsilon(z_1 - \mathbf{p}_1)). \end{cases} \tag{3.10}$$

Under this transformation, the Laplace operator and the boundary derivative operator become

$$\Delta_z = \Delta_y + (\rho'(\varepsilon y_1))^2\partial_{y_2y_2} - 2\rho'(\varepsilon y_1)\partial_{y_1y_2} - \varepsilon\rho''(\varepsilon y_1)\partial_{y_2},$$

$$(1 + \rho'(\varepsilon y_1)^2)^{\frac{1}{2}}\frac{\partial}{\partial \nu} = \rho'(\varepsilon y_1)\partial_{y_1} - (1 + \rho'(\varepsilon y_1)^2)\partial_{y_2}.$$

Let $v^{(1)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \frac{w'}{|y|}\frac{\rho''(0)}{2}y_1^2 \text{ on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.11}$$

where $\mathbb{R}^2_+$ is the upper half plane, namely $\mathbb{R}^2_+ = \{y = (y_1, y_2) \in \mathbb{R}^2 | y_2 > 0\}$.

Let $v^{(2)}$ be the unique solution of

$$\begin{cases} \Delta v - v - 2\rho''(0)y_1\frac{\partial^2 v^{(1)}}{\partial_{y_1}\partial_{y_2}} = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = -\rho''(0)y_1\frac{\partial v^{(1)}}{\partial y_1} \text{ on } \partial\mathbb{R}^2_+. \end{cases}$$

Let $v^{(3)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \frac{w'}{|y|}\frac{1}{3}\rho^{(3)}(0)y_1^3 \text{ on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.12}$$

Note that $v^{(1)}, v^{(2)}$ are even functions in y_1 and $v^{(3)}$ is an odd function in y_1. Moreover, it is easy to see that $|v_i(y)| \leq Ce^{-\mu|y|}$ for any $0 < \mu < 1$. Let χ be a smooth cut-off function, such that $\chi(a) = 1$ for $a \in B(0, R_0\sqrt{D}\log\frac{\xi_\sigma}{\varepsilon D})$, and $\chi(a) = 0$ for $x \in B(0, 2R_0\sqrt{D}\log\frac{\xi_\sigma}{\varepsilon D})^c$ for some suitable R_0 such that $|p_i - p^0| < \frac{R_0}{\sigma}\log\frac{\xi_\sigma}{\epsilon D}$, and

$$\chi_\varepsilon(z - p) = \chi(\varepsilon(z - p)) \quad \text{ for } z \in \Omega_\varepsilon. \tag{3.13}$$

Set

$$h_p(z) = -(\varepsilon v^{(1)}(y) + \varepsilon^2 (v^{(2)}(y) + v^{(3)}(y)))\chi_\varepsilon(z-p) + \varepsilon^3 \xi_p(z),\ z \in \Omega_\varepsilon. \tag{3.14}$$

Then we have the following estimate:

Proposition 3.1.

$$\|\xi_p(z)\|_{H^1(\Omega_\varepsilon)} \le C. \tag{3.15}$$

Proposition 3.1 was proved in [21] by Taylor expansion including a rigorous bound for the remainder using estimates for elliptic partial differential equations. Moreover, it has been shown that $|\xi_p(z)| \le Ce^{-\mu|z-p|}$ for any $0 < \mu < 1$.

Similarly we know from [21] that

Proposition 3.2.

$$[\frac{\partial w}{\partial \tau(p)} - \frac{\partial p_{\Omega_\varepsilon} w}{\partial \tau(p)}](z-p) = \varepsilon \eta(y)\chi_\varepsilon(z-p) + \varepsilon^2 \eta_1(z),\ z \in \Omega_\varepsilon, \tag{3.16}$$

where η is the unique solution of the following equation:

$$\begin{cases} \Delta\eta - \eta = 0 \ \ in\ \mathbb{R}^2_+, \\ \frac{\partial \eta}{\partial y_2} = -\frac{1}{2}(\frac{w''}{|y|^2} - \frac{w'}{|y|^3})\rho''(0)y_1^3 - \frac{w'}{|y|}\rho''(0)y_1 \ \ on\ \partial R^2_+. \end{cases} \tag{3.17}$$

Moreover,

$$\|\eta_1\|_{H^1(\Omega_\varepsilon)} \le C. \tag{3.18}$$

It follows that $\eta(y)$ is an odd function in y_1. It can be seen that $|\eta_1(y)| \le Ce^{-\mu|y|}$ for some $0 < \mu < 1$.

Finally, let

$$L_0 = \Delta - 1 + 2w(z). \tag{3.19}$$

We have

Lemma 3.1.

$$Ker(L_0) \cap H^2_N(\mathbb{R}^2_+) = span\{\frac{\partial w}{\partial y_1}\}, \tag{3.20}$$

where $H^2_N(\mathbb{R}^2_+) = \{u \in H^2(\mathbb{R}^2_+) : \frac{\partial u}{\partial y_2} = 0 \ on\ \partial\mathbb{R}^2_+\}$.

Proof. See Lemma 4.2 in [14]. □

Remark 3.2. In the following sections, we will denote by $y^i = (y_1^i, y_2^i)$ the transformation defined by (3.10) centred at the point p_i and let $v_i^{(j)}$ be the corresponding solutions in the expansion of h_{p_i}.

3.2. The analysis of the Green's function

Next we introduce a Green's function $G_{\sqrt{D}}$ which is needed to derive our main results.

For $D > 0$, let $G_D(x, y)$ be the Green's function given by

$$\begin{cases} -\Delta G_{\sqrt{D}} + G_{\sqrt{D}} = \delta_y \text{ in } \Omega_{\sqrt{D}}, \\ \frac{\partial G_{\sqrt{D}}}{\partial \nu} = 0 \text{ on } \partial\Omega_{\sqrt{D}}, \end{cases} \tag{3.21}$$

where $y \in \partial\Omega_{\sqrt{D}}$ and let G_0 be the Green's function of the upper half plane:

$$-\Delta G_0 + G_0 = \delta_0 \text{ in } \mathbb{R}^2_+, \frac{\partial G_0}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+. \tag{3.22}$$

Then $H(x) = G_{\sqrt{D}}(x) - G_0(x)$ will satisfy

$$\begin{cases} \Delta H - H = 0 \text{ in } \Omega_{\sqrt{D}}, \\ \frac{\partial H}{\partial \nu} = -\frac{\partial G_0}{\partial \nu} \text{ on } \partial\Omega_{\sqrt{D}}. \end{cases} \tag{3.23}$$

Let η_1 be the solution of

$$\begin{cases} \Delta\eta_1 - \eta_1 = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \eta_1}{\partial y_2} = -\frac{1}{2}\sqrt{D}\frac{G_0'(|y|)}{|y|}\rho''(0)y_1^2, \end{cases} \tag{3.24}$$

and let η_2 be the solution of

$$\begin{cases} \Delta\eta_2 - \eta_2 = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \eta_2}{\partial y_2} = D(-\frac{1}{3}\frac{G_0'(|y|)}{|y|}\rho^{(3)}(0)y_1^3) \text{ on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.25}$$

It can be seen easily that η_1 is even in y_1 and η_2 is odd in y_1. Then one can get the following result.

Lemma 3.2.

$$G_{\sqrt{D}}(x, p) = G_0(x, p) + \sqrt{D}\eta_1(y)\chi_{\sqrt{D}}(x - p) + D\eta_2(y)\chi_{\sqrt{D}}(x - p) + O(D^{\frac{3}{2}}). \tag{3.26}$$

Proof. First we compute on $\partial\Omega_{\sqrt{D}}$,

$$\begin{aligned} &\sqrt{1 + \rho'(\sqrt{D}x_1)^2}\frac{\partial}{\partial\nu}G_0(x) \\ &= \frac{G_0'(|x|)}{|x|}(x_2 - x_1\rho'(\sqrt{D}x_1)) \\ &= \frac{G_0'(|x|)}{|x|}(-\frac{1}{2}\sqrt{D}\rho''(0)y_1^2 - \frac{1}{3}D\rho^{(3)}(0)y_1^3) + O(D^{\frac{3}{2}}e^{-l|y|}), \end{aligned}$$

for any $0 < l < 1$.

Since we have

$$\frac{G_0'(|x|)}{|x|} = \frac{G_0'(|y|)}{|y|} + D\frac{|y|G_0''(|y|) - G_0'(|y|)}{8|y|^3}(\rho''(0)y_1^2)^2 + O(D^{\frac{3}{2}}e^{-l|y|}),$$

we get

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$$\sqrt{1+\rho'(\sqrt{D}x_1)^2}\frac{\partial}{\partial\nu}G_0(x)$$
$$=-\frac{1}{2}\sqrt{D}\frac{G_0'(|y|)}{|y|}\rho''(0)y_1^2-\frac{1}{3}D\frac{G_0'(|y|)}{|y|}\rho^{(3)}(0)y_1^3+O(D^{\frac{3}{2}}e^{-l|y|}).$$

From the expansion above, we can get the asymptotic behaviour of $G_{\sqrt{D}}$. □

Next we have the following expansion of G_0:

Lemma 3.3. *The following expansion of G_0 holds:*

$$G_0(r)=-\frac{1}{\pi}\log r+c_1+c_2r^2\log r+\psi(r),$$

for $0<r<1$, where ψ is a smooth function with $\psi(0)=\psi'(0)=0$ and c_1,c_2 are universal constants.

Proof. By an even extension in y_2, one can get the Green's function in the whole space $\mathbb{R}^2$. For the expansion of fundamental solution, see Lemma 4.1 in [3]. Then the above expansion follows. □

We set

$$G_{\sqrt{D}}(x,y)=\frac{1}{\pi}\log\frac{1}{|x-y|}+\tilde{H}(x,y). \tag{3.27}$$

From the estimates above, and for points $\mathbf{p}^\varepsilon\in\Lambda_k$, we have

$$G_{\sqrt{D}}(\sigma p_i,\sigma p_j)=O(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})\text{ for }|i-j|=1, \tag{3.28}$$
$$G_{\sqrt{D}}(\sigma p_i,\sigma p_j)=O((\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2)\text{ for }|i-j|=2. \tag{3.29}$$

Generally, we have

$$G_{\sqrt{D}}(\sigma p_i,\sigma p_j)=O((\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^{|i-j|})\text{ for }|i-j|\geq 1. \tag{3.30}$$

For the derivatives, we estimate

$$\frac{\partial^l}{\partial p_i^l}G_{\sqrt{D}}(\sigma p_i,\sigma p_j)=O((\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^{|i-j|}\sigma^l)\text{ for }|i-j|\geq 1. \tag{3.31}$$

3.3. Calculating the heights of the spikes

In this section, we are going to determine the heights of spikes $\xi_{\sigma,i}$ to leading order. In the sequel, by $T[h]$ we denote the unique solution of the equation

$$\begin{cases}\Delta v-\sigma^2v+h=0\text{ in }\Omega_\varepsilon,\\ \frac{\partial v}{\partial\nu}=0\text{ on }\partial\Omega_\varepsilon.\end{cases} \tag{3.32}$$

Then we know that

$$v(z)=\int\limits_{\Omega_\varepsilon}G_{\sqrt{D}}(\sigma z,\sigma x)h(x)\,dx. \tag{3.33}$$

As mentioned before, we will choose the approximate solution to be

$$U(z)=\sum_{i=1}^{k}\xi_{\sigma,i}Pw_{p_i}(z-p_i) \tag{3.34}$$

and

$$\begin{aligned}V(z)&=T[U^2](z) \\ &=\int\limits_{\Omega_\varepsilon}G_{\sqrt{D}}(\sigma z,\sigma x)\left(\sum_{i=1}^{k}\xi_{\sigma,i}Pw_{p_i}(x-p_i)\right)^2dx.\end{aligned} \tag{3.35}$$

First we calculate the heights of the peaks:

$$\begin{aligned}V(p_i)&=\int\limits_{\Omega_\varepsilon}G_{\sqrt{D}}(\sigma p_i,\sigma x)\left(\sum_{i=1}^{k}\xi_{\sigma,i}Pw_{p_i}(x-p_i)\right)^2dx \\ &=\int\limits_{\Omega_\varepsilon}G_{\sqrt{D}}(\sigma p_i,\sigma x)\left(\sum_{i=1}^{k}\xi_{\sigma,i}^2\left(Pw_{p_i}(x-p_i)\right)^2\right)dx+O(\varepsilon^4) \\ &=\xi_{\sigma,i}^2\int\limits_{\Omega_\varepsilon}\left(\frac{1}{\pi}\log\frac{1}{\sigma|x-p_i|}+\tilde H(\sigma x,\sigma p_i)\right)\left(Pw_{p_i}(x-p_i)\right)^2dx \\ &+\sum_{j\neq i}\xi_{\sigma,j}^2\int\limits_{\Omega_\varepsilon}G_{\sqrt{D}}(\sigma x,\sigma p_i)(Pw_{p_j}(x-p_j))^2\,dx+O(\varepsilon^4) \\ &=(\frac{1}{\pi}\log\frac{1}{\sigma}\int\limits_{\mathbb{R}^2_+}w^2\,dx)\xi_{\sigma,i}^2+\sum_{j=1}^{k}O(\xi_{\sigma,j}^2).\end{aligned}$$

Thus

$$\xi_{\sigma,i}=(\frac{1}{\pi}\log\frac{1}{\sigma}\int\limits_{\mathbb{R}^2_+}w^2\,dx)\xi_{\sigma,i}^2+\sum_{j=1}^{k}O(\xi_{\sigma,j}^2). \tag{3.36}$$

We assume that the heights of the spikes are asymptotically equal as $\varepsilon, D\to 0$, i.e.

$$\lim_{\sigma\to 0}\frac{\xi_{\sigma,i}}{\xi_{\sigma,j}}=1,\ \text{ for } i\neq j. \tag{3.37}$$

Then we get that

$$\begin{aligned}\xi_{\sigma,i}&=(\frac{1}{\pi}\log\frac{1}{\sigma}\int\limits_{\mathbb{R}^2_+}w^2\,dx)^{-1}(1+O(\frac{1}{\log\frac{1}{\sigma}})) \\ &=\xi_\sigma(1+O(\frac{1}{\log\frac{1}{\sigma}})),\end{aligned} \tag{3.38}$$

where

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$$\xi_\sigma = \left(\frac{1}{\pi}\log\frac{1}{\sigma}\int\limits_{\mathbb{R}^2_+} w^2\,dx\right)^{-1}. \tag{3.39}$$

The analysis in this subsection calculates the heights of the spikes under the assumption that their shape is given. In the next two sections, we provide the rigorous proof for the existence.

4. Existence I: reduction to finite dimensions

Let us start to prove Theorem 2.1.

The first step is choosing a good approximate solution which was done in (3.6). The second step is using the Liapunov–Schmidt method to reduce the problem to finite dimensions which we do in this section. The last step is solving the reduced problem which will be done in Section 5.

First we need to calculate the error terms caused by the approximate solution given in (3.6) to show that this is a good choice:

$$\begin{aligned} S_1(U,V) &= \Delta U - U + \frac{U^2}{V} \\ &= \frac{U^2}{V} - \sum_{i=1}^k \xi_{\sigma,i} w(x-p_i)^2 \\ &= \frac{\sum_{i=1}^k \xi_{\sigma,i}^2 Pw_{p_i}(x-p_i)^2}{V} - \sum_{i=1}^k \xi_{\sigma,i} w(x-p_i)^2 + O(\varepsilon^4) \\ &= \sum_{i=1}^k \xi_{\sigma,i}(Pw_i^2 - w_i^2) + \sum_{i=1}^k \xi_{\sigma,i}^2 Pw_i^2\left(\frac{1}{V(x)} - \frac{1}{V(p_i)}\right) + O(\varepsilon^4), \end{aligned}$$

where we have used the notation

$$Pw_i(x) = Pw_{p_i}(x-p_i),\;\; w_i(x) = w(x-p_i). \tag{4.1}$$

On the other hand, we calculate for $x = p_i + z$

$$\begin{aligned} Pw_i(x)^2 - w_i^2(x) \;&=\; 2w(z)(\varepsilon v_i^{(1)}(z) + \varepsilon^2 v_i^{(2)}(z) + \varepsilon^2 v_i^{(3)})(z) + \varepsilon^2 (v_i^{(1)}(z))^2 + O(\varepsilon^3) \\ &:= \varepsilon R_{1,i}(z) + \varepsilon^2 R_{2,i}(z) + O(\varepsilon^3), \end{aligned}$$

where $R_{1,i}(z) = 2w(z)[v_i^{(1)}(z) + \varepsilon v_i^{(2)}(z)] + \varepsilon (v_1^{(i)}(z))^2$, $R_{2,i} = 2w(z)v_i^{(3)}(z)$. This implies

$$\begin{aligned} &V(p_i+z) - V(p_i) \\ &= \int\limits_{\Omega_\varepsilon} [G_{\sqrt{D}}(\sigma p_i + \sigma z, \sigma x) - G_{\sqrt{D}}(\sigma p_i, \sigma x)]\Big(\sum_{i=1}^k \xi_{\sigma,i} Pw_i\Big)^2\,dx \\ &= \int\limits_{\Omega_\varepsilon} \frac{1}{\pi}\log\frac{|x-p_i|}{|x-z-p_i|} Pw_i^2 \xi_{\sigma,i}^2 + (\tilde{H}(\sigma p_i + \sigma y, \sigma x) - \tilde{H}(\sigma p_i, \sigma x)) Pw_i^2 \xi_{\sigma,i}^2\,dx \\ &+ \sum_{j\neq i} \xi_{\sigma,j}^2 \int\limits_{\Omega_\varepsilon} [G_{\sqrt{D}}(\sigma p_i + \sigma z, \sigma x) - G_{\sqrt{D}}(\sigma z, \sigma x)] Pw_j^2\,dx + O(\varepsilon^4) \\ &= \xi_{\sigma,i}^2 \int\limits_{\mathbb{R}^2_+} \log\frac{|y|}{|y-z|} w^2(y)(1+o(1))\,dy + \sum_{j\neq i} \xi_{\sigma,j}^2 \nabla_{p_i} G_{\sqrt{D}}(\sigma p_i, \sigma p_j)\cdot z \int\limits_{\mathbb{R}^2_+} w^2\,dy \end{aligned}$$

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$$+O(\xi_\sigma^2\varepsilon^2 h'(\varepsilon p_i)) + O(\xi_\sigma^2\varepsilon^3) + O(\sum_{j\neq i}\xi_\sigma^2|\nabla_{p_i}^2 G_{\sqrt{D}}(\sigma p_i,\sigma p_j)|)$$
$$:= \xi_{\sigma,i}^2 R_1(z) + \xi_{\sigma,i}^2 R_2(z) + h.o.t,$$

where $R_1(z)$ is even in z_1 and $R_2(z)$ is odd in z_1 and

$$R_1(z) = O(\log(1+|z|)),\ R_2(z) = O(\sum_{j\neq i}|\nabla_{p_i} G_{\sqrt{D}}(\sigma p_i,\sigma p_j)||z|) \tag{4.2}$$

and

$$\begin{aligned} h.o.t\ &= O(\xi_\sigma^2\varepsilon^2 h'(\varepsilon p_i)) + O(\sum_{i\neq j}\xi_\sigma^2|\nabla_{p_i}^2 G_{\sqrt{D}}(\sigma p_i,\sigma p_j)|) \\ &= O(\xi_\sigma^2\varepsilon^2\sqrt{D}\log\frac{\xi_\sigma}{\varepsilon D}). \end{aligned} \tag{4.3}$$

Thus we can get that

$$\frac{1}{V(p_i+z)} - \frac{1}{V(p_i)} = \frac{1}{V(p_i)^2}(-\xi_{\sigma,i}^2 R_1(z) - \xi_{\sigma,i}^2 R_2(z) + h.o.t). \tag{4.4}$$

By the above estimates, we have the following key estimate:

Lemma 4.1. *For $x = p_i + z$, $|z| < \frac{R_0}{\sigma}\log\frac{\varepsilon D}{\xi_\sigma}$, we have*

$$S_1(U,V) = S_{1,1} + S_{1,2}, \tag{4.5}$$

where

$$S_{1,1} = \xi_{\sigma,i}^2\tilde{R}_1(z) \tag{4.6}$$
$$S_{1,2} = -\xi_{\sigma,i}^2 w^2(z)R_2(z) + \xi_{\sigma,i}\varepsilon^2 R_{2,i}(z) + h.o.t, \tag{4.7}$$

where $\tilde{R}_1(z)$ is even in z_1 with the property that $\tilde{R}_1(z) = O(\log(1+|z|))$, and $R_2(z), R_{2,i}(z)$ are defined above.

Further, $S_1(U,V) = \varepsilon^{\frac{R_0}{\sigma}}$ for $|x-p_i| \geq \frac{R_0}{\sigma}\log\frac{\varepsilon D}{\xi_\sigma}$ for all i.

The above estimates will be very useful in the existence proof using the Liapunov–Schmidt reduction. In particular, they will imply an explicit formula for the positions of the spikes in Section 5.

Now we study the linearised operator defined by

$$L_{\varepsilon,\mathbf{p}} := S'\begin{pmatrix} U \\ V \end{pmatrix}, \tag{4.8}$$

$$L_{\varepsilon,\mathbf{p}} : H_N^2(\Omega_\varepsilon)\times H_N^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)\times L^2(\Omega_\varepsilon). \tag{4.9}$$

We first define

$$K_{\varepsilon,\mathbf{p}} = C_{\varepsilon,\mathbf{p}} = \mathrm{Span}\{\frac{\partial U}{\partial\tau(p_i)}, i = 1,\cdots,k\} \tag{4.10}$$

and define the approximate kernels by

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$$\mathcal{K}_{\varepsilon,\mathbf{p}} := K_{\varepsilon,\mathbf{p}} + \{0\} \subset H^2_N(\Omega_\varepsilon) \times H^2_N(\Omega_\varepsilon),$$

and choose the approximate cokernels as follows:

$$\mathcal{C}_{\varepsilon,\mathbf{p}} := C_{\varepsilon,\mathbf{p}} + \{0\} \subset L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon).$$

We then define

$$\mathcal{K}^\perp_{\varepsilon,\mathbf{p}} := K^\perp_{\varepsilon,\mathbf{p}} + H^2_N(\Omega_\varepsilon) \subset H^2_N(\Omega_\varepsilon) \times H^2_N(\Omega_\varepsilon),$$
$$\mathcal{C}^\perp_{\varepsilon,\mathbf{p}} := C^\perp_{\varepsilon,\mathbf{p}} + L^2(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon),$$

where $C^\perp_{\varepsilon,\mathbf{p}}, K^\perp_{\varepsilon,\mathbf{p}}$ denote the orthogonal complements with the scalar product of $L^2(\Omega_\varepsilon)$.

Let $\pi_{\varepsilon,\mathbf{p}}$ denote the projection in $L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ onto $\mathcal{C}^\perp_{\varepsilon,\mathbf{p}}$. We are going to show that the equation

$$\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} U+\phi \\ V+\psi \end{pmatrix} = 0$$

has a unique solution $\Sigma_{\varepsilon,\mathbf{p}} = \begin{pmatrix} \phi_{\varepsilon,\mathbf{p}} \\ \psi_{\varepsilon,\mathbf{p}} \end{pmatrix} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$ if $\max\{\sigma, D\}$ is small enough.

Set

$$\mathcal{L}_{\varepsilon,\mathbf{p}} = \pi_{\varepsilon,\mathbf{p}} \circ L_{\varepsilon,\mathbf{p}} : \mathcal{K}^\perp_{\varepsilon,\mathbf{p}} \to \mathcal{C}^\perp_{\varepsilon,\mathbf{p}}. \tag{4.11}$$

Written in components, we have

$$\mathcal{L}_{\varepsilon,\mathbf{p}} := \begin{pmatrix} \mathcal{L}_{\varepsilon,\mathbf{p},1} \\ \mathcal{L}_{\varepsilon,\mathbf{p},2} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathcal{L}_{\varepsilon,\mathbf{p},1} \\ \mathcal{L}_{\varepsilon,\mathbf{p},2} \end{pmatrix} \begin{pmatrix} \phi_{\varepsilon,\mathbf{p}} \\ \psi_{\varepsilon,\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \Delta\phi_{\varepsilon,\mathbf{p}} - \phi_{\varepsilon,\mathbf{p}} + \frac{2U}{V}\phi_{\varepsilon,\mathbf{p}} - \frac{U^2}{V^2}\psi_{\varepsilon,\mathbf{p}} \\ \Delta\psi_{\varepsilon,\mathbf{p}} - \psi_{\varepsilon,\mathbf{p}} + 2U\phi_{\varepsilon,\mathbf{p}} \end{pmatrix}.$$

As a preparation we state a result on the invertibility of the corresponding linearised operator $\mathcal{L}_{\varepsilon,\mathbf{p}}$ whose proof is postponed to Appendix A.

Proposition 4.1. *There exist positive constants $\bar{\delta}, C$ such that for $\max\{\sigma, D\} < \bar{\delta}$, the map $\mathcal{L}_{\varepsilon,\mathbf{p}}$ is surjective for arbitrary $\mathbf{p} \in \Lambda_k$. Moreover the following estimate holds:*

$$\|\Sigma_{\varepsilon,\mathbf{p}}\|_{H^2(\Omega_\varepsilon)\times H^2(\Omega_\varepsilon)} \le C(\|\mathcal{L}_{\varepsilon,\mathbf{p},1}(\Sigma_{\varepsilon,\mathbf{p}})\|_{H^2(\Omega_\varepsilon)} + \xi_\sigma^{-1}\|\mathcal{L}_{\varepsilon,\mathbf{p},2}(\Sigma_{\varepsilon,\mathbf{p}})\|_{H^2(\Omega_\varepsilon)}). \tag{4.12}$$

Now we are in the position to solve the equation

$$\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} U+\phi \\ V+\psi \end{pmatrix} = 0. \tag{4.13}$$

Since $\mathcal{L}_{\varepsilon,\mathbf{p}}|_{\mathcal{K}^\perp_{\varepsilon,\mathbf{p}}}$ is invertible, we can write the above equation as

$$\Sigma = -\mathcal{L}^{-1}_{\varepsilon,\mathbf{p}} \circ \pi_{\varepsilon,\mathbf{p}}(S_\varepsilon \begin{pmatrix} U \\ V \end{pmatrix}) - \mathcal{L}^{-1}_{\varepsilon,\mathbf{p}} \circ \pi_{\varepsilon,\mathbf{p}}(N_{\varepsilon,\mathbf{p}}(\Sigma)) := M_{\varepsilon,\mathbf{p}}(\Sigma), \tag{4.14}$$

where

$$\Sigma = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

and

$$N_{\varepsilon,\mathbf{p}}(\Sigma) = S_\varepsilon \begin{pmatrix} U+\phi \\ V+\psi \end{pmatrix} - S_\varepsilon \begin{pmatrix} U \\ V \end{pmatrix} - S'_\varepsilon \begin{pmatrix} U \\ V \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

and the operator $M_{\varepsilon,\mathbf{p}}$ is defined above for $\Sigma \in H^2_N(\Omega_\varepsilon) \times H^2_N(\Omega_\varepsilon)$. We are going to show that the operator $M_{\varepsilon,\mathbf{p}}$ is a contraction mapping on

$$B_\varepsilon = \{\Sigma \in H^2_N(\Omega_\varepsilon) \times H^2_N(\Omega_\varepsilon) | \|\Sigma\|_{H^2 \times H^2} < C_0 \xi_\sigma^2\} \tag{4.15}$$

if C_0 is large enough. We have that

$$\begin{aligned} \|M_{\varepsilon,\mathbf{p}}(\Sigma)\|_{H^2\times H^2} &\le C(\|\pi_{\varepsilon,\mathbf{p}} \circ N_{\varepsilon,\mathbf{p},1}(\Sigma)\|_{L^2} + \xi_\sigma^{-1} \|\pi_{\varepsilon,\mathbf{p}} \circ N_{\varepsilon,\mathbf{p},2}(\Sigma)\|_{L^2} \\ &\quad + \|\pi_{\varepsilon,\mathbf{p}} \circ S_\varepsilon \begin{pmatrix} U \\ V \end{pmatrix} \|_{L^2\times L^2} \\ &\le C(c(\xi_\sigma)\xi_\sigma^2 + \xi_\sigma^2), \end{aligned}$$

where $C > 0$ is independent of $\varepsilon > 0$ and $c(\xi_\sigma) \to 0$ as $\xi_\sigma \to 0$. Similarly we can show that

$$\|M_{\varepsilon,\mathbf{p}}(\Sigma) - M_{\varepsilon,\mathbf{p}}(\Sigma')\|_{H^2\times H^2} \le Cc(\xi_\sigma)\|\Sigma - \Sigma'\|_{H^2\times H^2},$$

where $c(\xi_\sigma) \to 0$ as $\xi_\sigma \to 0$. If we choose C_0 large enough, then $M_{\varepsilon,\mathbf{p}}$ is a contraction mapping on B_ε. The existence of a fixed point $\Sigma_{\varepsilon,\mathbf{p}}$ together with an error estimate now follows from the contraction mapping principle. Moreover, $\Sigma_{\varepsilon,\mathbf{p}}$ is a solution. Thus we have proved

Lemma 4.2. *There exists $\bar\delta > 0$ such that for every triple $(\varepsilon, D, \mathbf{p})$ with $\max\{\sigma, D\} < \bar\delta$, and $\mathbf{p} \in \Lambda_k$, there exists a unique $(\phi_{\varepsilon,\mathbf{p}}, \psi_{\varepsilon,\mathbf{p}}) \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$ satisfying*

$$S_\varepsilon \begin{pmatrix} U+\phi_{\varepsilon,\mathbf{p}} \\ V+\psi_{\varepsilon,\mathbf{p}} \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{p}}, \tag{4.16}$$

and

$$\|(\phi_{\varepsilon,\mathbf{p}}, \psi_{\varepsilon,\mathbf{p}})\|_{H^2\times H^2} \le C\xi_\sigma^2.$$

More refined estimates for $\phi_{\varepsilon,\mathbf{p}}$ are needed. We recall that from Lemma 4.1 that $S_1(U,V)$ can be decomposed into two parts $S_{1,1}$ and $S_{1,2}$, where $S_{1,1}$ is in leading order an even function in z_1 and $S_{1,2}$ is in leading order an odd function in z_1. Similarly we can decompose $\phi_{\varepsilon,\mathbf{p}}$.

Lemma 4.3. *Let $\phi_{\varepsilon,\mathbf{p}}$ be defined by (4.16). Then for $x = p_i + z$, we have*

$$\phi_{\varepsilon,\mathbf{p}}(x) = \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}, \tag{4.17}$$

where $\phi_{\varepsilon,\mathbf{p},1}$ is even in z_1 which can be estimated by

$$\phi_{\varepsilon,\mathbf{p},1} = O(\xi_\sigma^2) \ in \ H_N^2(\Omega_\varepsilon), \tag{4.18}$$

and $\phi_{\varepsilon,\mathbf{p},2}$ *can be estimated by*

$$\begin{aligned} \phi_{\varepsilon,\mathbf{p},2} &= O(\sum_{j\neq i} \xi_\sigma^2 \sigma |\nabla G_{\sqrt{D}}(\sigma p_i, \sigma p_j)|) + O(\sum_{i=1}^k \xi_\sigma \varepsilon^2 h'(\varepsilon p_i)) \\ &= O(\xi_\sigma \varepsilon^2 \sqrt{D} \log \frac{\xi_\sigma}{\varepsilon D}). \end{aligned} \tag{4.19}$$

Proof. Let $S(u) = S_1(u, T(u))$. We first solve

$$S(U + \phi_{\varepsilon,\mathbf{p},1}) - S(U) + \sum_{i=1}^k S_{1,1}(x - p_i) \in C_{\varepsilon,\mathbf{p}}, \tag{4.20}$$

for $\phi_{\varepsilon,\mathbf{p},1} \in K_{\varepsilon,\mathbf{p}}^\perp$. Then we solve

$$S[U + \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}] - S[U + \phi_{\varepsilon,\mathbf{p},1}] + \sum_{i=1}^k S_{1,2}(x - p_i) \in C_{\varepsilon,\mathbf{p}}$$

for $\phi_{\varepsilon,\mathbf{p},2} \in K_{\varepsilon,\mathbf{p}}^\perp$.

Using the same proof as in Lemma 4.2, the above two equations are uniquely solvable for $\max\{\sigma, D\} \ll 1$. By uniqueness, $\phi_{\varepsilon,\mathbf{p}} = \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2}$. Since $S_{1,2} = S_{1,2}^0 + S_{1,2}^\perp$, where $S_{1,2}^0 = O(\xi_\sigma \varepsilon^2 \sqrt{D} \log \frac{\xi_\sigma}{\varepsilon D})$ and $S_{1,2}^\perp \in C_{\varepsilon,\mathbf{p}}^\perp$, it is easy to see that $\phi_{\varepsilon,\mathbf{p},1}$ and $\phi_{\varepsilon,\mathbf{p},2}$ have the required properties. □

5. Existence proof II: the reduced problem

In this section, we solve the reduced problem. This completes the proof for our main existence result given in Theorem 2.1.

By Lemma 4.2, for every $\mathbf{p} \in \Lambda_k$, there exists a unique solution $(\phi, \psi) \in \mathcal{K}_{\varepsilon,\mathbf{p}}^\perp$ such that

$$S_\varepsilon \begin{pmatrix} U + \phi \\ V + \psi \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{p}}. \tag{5.1}$$

We need to determine $\mathbf{p} = (p_1, \cdots, p_k) \in \Lambda_k$ such that

$$S_\varepsilon \begin{pmatrix} U + \phi \\ V + \psi \end{pmatrix} \perp \mathcal{C}_{\varepsilon,p}$$

and therefore $S_\varepsilon \begin{pmatrix} U + \phi \\ V + \psi \end{pmatrix} = 0$.

To this end, we calculate the projection:

$$\begin{aligned} &\int_{\Omega_\varepsilon} S_1(U + \phi, V + \psi) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \\ &= \int_{\Omega_\varepsilon} \Big(\Delta(U + \phi) - (U + \phi) + \frac{(U + \phi)^2}{V + \psi} \Big) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\Omega_\varepsilon} \Big(\Delta(U+\phi) - (U+\phi) + \frac{(U+\phi)^2}{V}\Big)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&+ \int\limits_{\Omega_\varepsilon} \Big(\frac{(U+\phi)^2}{V+\psi} - \frac{(U+\phi)^2}{V}\Big)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&= I_1 + I_2,
\end{aligned}
$$

where I_1, I_2 are defined by the last equality.

For I_1, we have

$$
\begin{aligned}
I_1 &= \int\limits_{\Omega_\varepsilon} \Big(\Delta(U+\phi) - (U+\phi) + \frac{(U+\phi)^2}{V}\Big)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&= \int\limits_{\Omega_\varepsilon} \Big(\Delta(\xi_{\sigma,i}Pw_i+\phi) - (\xi_{\sigma,i}Pw_i+\phi) + \frac{(\xi_{\sigma,i}Pw_i+\phi)^2}{V(p_i)}\Big)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&+ \int\limits_{\Omega_\varepsilon} (\xi_{\sigma,i}Pw_i+\phi)^2\big(\frac{1}{V(x)} - \frac{1}{V(p_i)}\big)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx + O(\varepsilon^4) \\
&= I_{11} + I_{12} + O(\varepsilon^4).
\end{aligned}
$$

Note that by the estimates satisfied by ϕ in Lemma 4.3, we have

$$
\begin{aligned}
&\int\limits_{\Omega_\varepsilon} (\Delta\phi - \phi + 2Pw_i\phi)\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&= \int\limits_{\Omega_\varepsilon} \phi\frac{\partial}{\partial\tau(p_i)}(Pw_i^2 - w_i^2) \\
&= \int\limits_{\Omega_\varepsilon} (\phi_{\varepsilon,\mathbf{p},1}+\phi_{\varepsilon,\mathbf{p},2})\frac{\partial}{\partial\tau(p_i)}[2\varepsilon w_i v_i^{(1)} + 2\varepsilon^2 w_i v_i^{(2)} + 2\varepsilon^2 w_i v_i^{(3)} + \varepsilon^2(v_i^{(1)})^2]\,dx + O(\xi_\sigma^2\varepsilon^3) \\
&= \sum_{j\neq i} O(\varepsilon\xi_\sigma^2|\nabla_{p_i}G_{\sqrt{D}}(\sigma p_i,\sigma p_j)|) + \sum_{i=1}^k \xi_\sigma^2\varepsilon^2|h'(\varepsilon p_i)| + O(\varepsilon^3\xi_\sigma^2) \\
&= O(\xi_\sigma)[\sum_{j\neq i}\xi_\sigma^2\sigma|\nabla G_{\sqrt{D}}(\sigma p_i,\sigma p_j)| + \sum_{i=1}^k \xi_\sigma\varepsilon^2|h'(\varepsilon p_i)|] \\
&= O(\xi_\sigma^2\sigma\varepsilon D\log\frac{\xi_\sigma}{\varepsilon D}),
\end{aligned}
\tag{5.2}
$$

where we have used the estimates (3.28)–(3.31). Further, we have

$$
\begin{aligned}
\int\limits_{\Omega_\varepsilon} \frac{\phi^2}{V(p_i)}\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx &= \frac{1}{\xi_{\sigma,i}}\int\limits_{\Omega_\varepsilon}(\phi_{\varepsilon,\mathbf{p},1}+\phi_{\varepsilon,\mathbf{p},2})^2\frac{\partial Pw_i}{\partial\tau(p_i)}\,dx \\
&= O(\xi_\sigma)(\sum_{j\neq i}\xi_\sigma^2\sigma|\nabla G_{\sqrt{D}}(\sigma p_i,\sigma p_j)| + \sum_{i=1}^k \xi_\sigma\varepsilon^2|h'(\varepsilon p_i)|) \\
&= O(\xi_\sigma^2\sigma\varepsilon D\log\frac{\xi_\sigma}{\varepsilon D}).
\end{aligned}
\tag{5.3}
$$

We compute

$$
\begin{aligned}
&\xi_{\sigma,i} \int_{\Omega_\varepsilon} (\Delta P w_i - P w_i + P w_i^2) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \\
&= \xi_{\sigma,i} \int_{\Omega_\varepsilon} (P w_i^2 - w_i^2) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \\
&= \xi_{\sigma,i} \int_{\mathbb{R}^2_+} \varepsilon (2 w v_i^{(1)} + 2\varepsilon w v_i^{(2)} + 2\varepsilon w v_i^{(3)} + \varepsilon (v_i^{(1)})^2) \qquad (5.4) \\
&\times (\frac{\partial w}{\partial y_1} + \varepsilon \frac{\partial v_i^{(1)}}{\partial y_1} + \varepsilon^2 \frac{\partial v_i^{(2)}}{\partial y_1} + \varepsilon^2 \frac{\partial v_i^{(3)}}{\partial y_1}) \, dy + O(\xi_\sigma \varepsilon^3) \\
&= 2\varepsilon^2 \xi_{\sigma,i} \int_{\mathbb{R}^2_+} w v_i^{(3)} \frac{\partial w}{\partial y_1} \, dy + O(\xi_\sigma^2 \sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \qquad (5.5)
\end{aligned}
$$

By (3.12), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2_+} 2w(y) \frac{\partial w(y)}{\partial y_1} v_i^{(3)} \, dy &= \int_{\mathbb{R}^2_+} -(\Delta - 1) \frac{\partial w(y)}{\partial y_1} v_i^{(3)} \, dy \\
&= -\int_{\mathbb{R}} \left(\frac{\partial w(y)}{\partial y_1} \frac{\partial v_i^{(3)}}{\partial y_2} - v_i^{(3)} \frac{\partial}{\partial y_2} \frac{\partial w(y)}{\partial y_1} \right) dy_1 \\
&= -\frac{1}{3} \int_{\mathbb{R}} (\frac{w'(|y|)}{|y|})^2 h'(\varepsilon p_i) y_1^4 \, dy_1 \\
&= -\nu_1 \frac{\partial}{\partial \tau(\varepsilon p_i)} h(\varepsilon p_i), \qquad (5.6)
\end{aligned}
$$

where the constant $\nu_1 > 0$ is defined by

$$
\nu_1 = \frac{1}{3} \int_{\mathbb{R}} \left(\frac{\partial w(y_1, 0)}{\partial y_1} \right)^2 y_1^2 \, dy_1 > 0. \qquad (5.7)
$$

Now by (5.2)–(5.6),

$$
I_{11} = -\varepsilon^2 \xi_\sigma \nu_1 \frac{\partial}{\partial \tau(\varepsilon p_i)} h(\varepsilon p_i) + O(\xi_\sigma^2 \sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \qquad (5.8)
$$

Next we estimate I_{12}:

$$
\begin{aligned}
I_{12} &= \int_{\Omega_\varepsilon} (\xi_{\sigma,i} P w_i + \phi)^2 (\frac{1}{V(x)} - \frac{1}{V(p_i)}) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \\
&= \int_{\Omega_\varepsilon} (\xi_{\sigma,i} P w_i + \phi_{\varepsilon,\mathbf{p},1} + \phi_{\varepsilon,\mathbf{p},2})^2 \frac{1}{V(p_i)^2} (-\xi_{\sigma,i}^2 R_1 - \xi_{\sigma,i}^2 R_2) \frac{\partial P w_i}{\partial \tau(p_i)} \, dx \\
&\quad + O(\xi_\sigma) (\sum_{j \neq i} \xi_\sigma^2 |\nabla_{p_i} G_{\sqrt{D}}(\sigma p_i, \sigma p_j)| + \sum_{i=1}^{k} \xi_\sigma \varepsilon^2 |h'(\varepsilon p_i)|)
\end{aligned}
$$

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$$
\begin{aligned}
&= -\xi_{\sigma,i}^2 \int_{\mathbb{R}_+^2} w(y)^2 R_2(y) \frac{\partial w}{\partial y_1}\, dy + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}) \\
&= -\sum_{j\neq i} \xi_\sigma^2 \frac{\partial G_{\sqrt{D}}(\sigma p_i, \sigma p_j)}{\partial \tau(p_i)} (\int_{\mathbb{R}_+^2} w^2 y_1 \frac{\partial w}{\partial y_1}\, dy \int_{\mathbb{R}_+^2} w^2\, dy) + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}) \\
&= \nu_2 \sum_{j\neq i} \xi_\sigma^2 \frac{\partial G_{\sqrt{D}}(\sigma p_i, \sigma p_j)}{\partial \tau(p_i)} + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}) \\
&= \nu_2 \sum_{j\neq i} \xi_\sigma^2 \frac{\partial G_0(\sigma p_i, \sigma p_j)}{\partial \tau(p_i)} + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}),
\end{aligned}
\tag{5.9}
$$

where we have used $\sqrt{D} \log \frac{\xi_\sigma}{\epsilon D} \ll 1$ and

$$
\nu_2 = \frac{1}{3} \int_{\mathbb{R}_+^2} w^3\, dy \int_{\mathbb{R}_+^2} w^2\, dy > 0. \tag{5.10}
$$

Thus by (5.8) and (5.9), we have

$$
\begin{aligned}
I_1 &= \nu_2 \sum_{j\neq i} \xi_\sigma^2 \frac{\partial G_0(\sigma p_i, \sigma p_j)}{\partial \tau(p_i)} - \varepsilon^2 \xi_\sigma \nu_1 \frac{\partial}{\partial \tau(\varepsilon p_i)} h(\varepsilon p_i) \\
&\quad + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}).
\end{aligned}
\tag{5.11}
$$

Next we estimate I_2:

$$
\begin{aligned}
I_2 &= \int_{\Omega_\varepsilon} (\frac{(U+\phi)^2}{V+\psi} - \frac{(U+\phi)^2}{V}) \frac{\partial P w_i}{\partial \tau(p_i)}\, dx \\
&= -\int_{\Omega_\varepsilon} \frac{(\xi_{\sigma,i} P w_i + \phi)^2}{V^2} \psi \frac{\partial P w_i}{\partial \tau(p_i)}\, dx \\
&\quad + O(\xi_\sigma)(\sum_{j\neq i} \xi_\sigma^2 |\nabla_{p_i} G_{\sqrt{D}}(\sigma p_i, \sigma p_j)| + \sum_{i=1}^k \xi_\sigma \varepsilon^2 |h'(\varepsilon p_i)|) \\
&= -\frac{1}{3} \int_{\Omega_\varepsilon} \frac{\partial (P w_i)^3}{\partial \tau(p_i)} (\psi(x) - \psi(p_i))\, dx + O(\xi_\sigma^2 \sigma\varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}).
\end{aligned}
$$

By the equation for ψ, we have

$$
\Delta \psi - \sigma^2 \psi + 2U\phi + \phi^2 = 0
$$

and therefore

$$
\begin{aligned}
\psi(p_i + z) - \psi(p_i) &= \int_{\Omega_\varepsilon} [G_{\sqrt{D}}(\sigma p_i, \sigma y) - G_{\sqrt{D}}(\sigma p_i + \sigma z, \sigma y)](2U\phi + \phi^2)(y)\, dy \\
&= O(\xi_\sigma \sum_{j\neq i} \xi_\sigma^2 |\nabla_{p_i} G_{\sqrt{D}}(\sigma p_i, \sigma p_j)||z|) + O(\xi_\sigma^3) R(z),
\end{aligned}
$$

where $R(z)$ is an even function in z_1.

Thus we have

$$I_2 = O(\xi_\sigma^2 \sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \tag{5.12}$$

Combining the estimates for I_1 and I_2, (5.11) and (5.12), we have

$$\begin{aligned} W_{\varepsilon,i} &:= \int\limits_{\Omega_\varepsilon} S_1[U+\phi, V+\psi] \frac{\partial P w_i}{\partial \tau(p_1)}\, dx \\ &= \xi_\sigma [\nu_2 \sum_{j \neq i} \xi_\sigma \sigma G_0'(\sigma|p_i - p_j|) \frac{p_i - p_j}{|p_i - p_j|} - \varepsilon^3 \nu_1 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_i - p^0)] \\ &+ o(\xi_\sigma \sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \end{aligned}$$

Thus $W_{\varepsilon,i} = 0$ is reduced to the following system:

$$\begin{cases} -\nu_2 \xi_\sigma \sigma G_0'(\sigma|p_1 - p_2|) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_1 - p^0) = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}), \\ \nu_2 \xi_\sigma \sigma (G_0'(\sigma|p_1 - p_2|) - G_0'(\sigma|p_2 - p_3|)) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_2 - p^0) = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}), \\ \cdots \\ \nu_2 \xi_\sigma \sigma (G_0'(\sigma|p_{k-1} - p_{k-2}|) - G_0'(\sigma|p_{k-1} - p_k|)) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_{k-1} - p^0) = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}), \\ \nu_2 \xi_\sigma \sigma G_0'(\sigma|p_k - p_{k-1}|) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_k - p^0) = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \end{cases} \tag{5.13}$$

We first solve the limiting case:

$$\begin{cases} -\nu_2 \xi_\sigma \sigma G_0'(\sigma|p_1^0 - p_2^0|) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_1^0 - p^0) = 0, \\ \nu_2 \xi_\sigma \sigma (G_0'(\sigma|p_1^0 - p_2^0|) - G_0'(\sigma|p_2^0 - p_3^0|)) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_2^0 - p^0) = 0, \\ \cdots \\ \nu_2 \xi_\sigma \sigma (G_0'(\sigma|p_{k-2}^0 - p_{k-1}^0|) - G_0'(\sigma|p_{k-1}^0 - p_k^0|)) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_{k-1}^0 - p^0) = 0, \\ \nu_2 \xi_\sigma \sigma G_0'(\sigma|p_{k-1}^0 - p_k^0|) - \nu_1 \varepsilon^3 \frac{\partial^2}{\partial \tau^2} h(\varepsilon p^0)(p_k^0 - p^0) = 0. \end{cases} \tag{5.14}$$

This system is uniquely solvable with

$$\sum_{i=1}^k p_i^0 = p^0. \tag{5.15}$$

Moreover, we have

$$\begin{aligned} \sigma(p_i^0 - p_{i-1}^0) &= \log \frac{\xi_\sigma}{\varepsilon D} - \frac{3}{2} \log \log \frac{\xi_\sigma}{\varepsilon D} \\ &- \log(-\frac{h''(p^0)\nu_1}{2\nu_2}) - \log[(i-1)(k+1-i)] + O(\frac{\log \log \frac{\xi_\sigma}{\varepsilon D}}{\log \frac{\xi_\sigma}{\varepsilon D}}), \end{aligned} \tag{5.16}$$

where we have used the notation $h''(p^0) = \frac{\partial^2}{\partial \tau^2} h(P^0) < 0$.

From (5.16), we know

$$\sum_{i=1}^{j} p_i^0 = O(\frac{1}{\sigma} \log \frac{\varepsilon D}{\xi_\sigma}) \text{ for } j = 1, \cdots, k-1. \tag{5.17}$$

To find p_i such that $W_{\varepsilon,i} = 0$, we expand $p_i = p_i^0 + \tilde{p}_i$. Then adding the first i equations, we have

$$-\nu_2 \xi_\sigma \sigma G_0''(\sigma |p_i^0 - p_{i+1}^0|)[\sigma(\tilde{p}_i - \tilde{p}_{i+1}) + O(\sigma^2 |\tilde{p}|^2)] + \nu_1 \sum_{j=1}^{i} \varepsilon^3 h''(p^0)\tilde{p}_i = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \tag{5.18}$$

One can get that

$$\tilde{p}_i = \tilde{p}_1(1 + o(1)) + O(\frac{1}{\sigma}), \text{ for } i = 2, \cdots, k.$$

By the last equation

$$\sum_{i=1}^{k} \nu_1 \varepsilon^3 h''(p^0)\tilde{p}_i = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}). \tag{5.19}$$

Thus

$$\nu_1 \varepsilon^3 h''(p^0) k \tilde{p}_1(1 + o(1)) = o(\sigma \varepsilon D \log \frac{\xi_\sigma}{\varepsilon D}) + O(\frac{\varepsilon^3}{\sigma}). \tag{5.20}$$

From the equation above, we can estimate $\tilde{p}_1$ by

$$\tilde{p}_1 = o(\frac{1}{\sigma} \log \frac{\varepsilon D}{\xi_\sigma}). \tag{5.21}$$

In conclusion, we solve $W_{\varepsilon,i} = 0$ with

$$\tilde{p}_i = o(\frac{1}{\sigma} \log \frac{\varepsilon D}{\xi_\sigma}).$$

Thus we have proved the following proposition:

Proposition 5.1. *For* $\max\{\sigma, D\}$ *small enough, there exists* $\mathbf{p}^\varepsilon \in \Lambda_k$ *with* $P_i \to P^0$ *such that* $W_{\varepsilon,i} = 0$.

Finally, we complete the proof of Theorem 2.1.

Proof. By Proposition 5.1, there exists $\mathbf{P}^\varepsilon \to \mathbf{P}^0$, such that $W_\varepsilon(\mathbf{p}^\varepsilon) = 0$. In other words, we have

$$S_\varepsilon \begin{pmatrix} U + \phi \\ V + \psi \end{pmatrix} = 0. \tag{5.22}$$

Moreover, by the maximum principle, $(U, V) > 0$ and the solution satisfies all the properties of Theorem 2.1. □

6. Study of the large eigenvalues

We consider the stability of the steady-state (u, v) constructed in Theorem 2.1.

In this section, we first study the large eigenvalues which satisfy $\lambda_\varepsilon \to \lambda_0 \neq 0$ as $\max\{\sigma, D\} \to 0$.

Linearizing the system around the equilibrium states (u, v) obtained in Theorem 2.1, we obtain the following eigenvalue problem:

$$\begin{cases} \Delta\phi - \phi + \frac{2u}{v}\phi - \frac{u^2}{v^2}\psi = \lambda\phi, \\ \Delta\psi - \sigma^2\psi + 2u\phi = \tau\lambda\sigma^2\psi \end{cases} \tag{6.1}$$

for $(\phi, \psi) \in H_N^2(\Omega_\varepsilon) \times H_N^2(\Omega_\varepsilon)$.

In this section, since we study the large eigenvalues, we may assume that $|\lambda_\varepsilon| \geq c > 0$ for $\max\{\sigma, D\}$ small enough. If $\mathrm{Re}(\lambda_\varepsilon) \leq -c < 0$, then λ_ε is a stable large eigenvalue, we are done. Therefore, we may assume that $\mathrm{Re}(\lambda_\varepsilon) \geq -c$ and for a subsequence $\max\{\sigma, D\} \to 0$, $\lambda_\varepsilon \to \lambda_0 \neq 0$. We shall derive the limiting eigenvalue problem which is given by a coupled system of NLEPs.

The second equation of (6.1) is equivalent to

$$\Delta\psi - \sigma^2(1 + \tau\lambda_\varepsilon)\psi + 2u\phi = 0. \tag{6.2}$$

We introduce the following notation:

$$\sigma_\lambda = \sigma\sqrt{1 + \tau\lambda_\varepsilon},$$

where in $\sqrt{1 + \tau\lambda_\varepsilon}$, we take the principal part of the square root.

Let us assume that

$$\|\phi\|_{H^2(\Omega_\varepsilon)} = 1.$$

We cut off ϕ as follows:

$$\phi_{\varepsilon,j} = \phi_\varepsilon\chi_\epsilon(z - p_j), \ j = 1, \cdots, k, \tag{6.3}$$

where the cutoff function χ_ϵ has been defined in (3.13).

From (6.1) and the exponential decay of w, it follows that

$$\phi_\varepsilon = \sum_{j=1}^{k} \phi_{\varepsilon,j} + O(\varepsilon^5). \tag{6.4}$$

Then by a standard procedure (see [8], Section 7.12), we extend $\phi_{\varepsilon,j}$ to a function defined on $\mathbb{R}^2$ such that

$$\|\phi_{\varepsilon,j}\|_{H^2(\mathbb{R}^2)} \leq C\|\phi_{\varepsilon,j}\|_{H^2(\Omega_\varepsilon)}, \ j = 1, \cdots, k.$$

Since $\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1$, $\|\phi_{\varepsilon,j}\|_{H^2(\mathbb{R}^2)} \leq C$. By taking a subsequence, we may assume that $\phi_{\varepsilon,j} \to \phi_j$ as $\max\{\sigma, D\} \to 0$ in $H^1(\mathbb{R}^2)$ for some $\phi_j \in H^1(\mathbb{R}^2)$ for $j = 1, \cdots, k$.

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By (6.1), we have

$$\begin{aligned}\psi_\varepsilon(p_j) &= \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma_\lambda p_j, \sigma_\lambda x) 2u\phi_\varepsilon(x)\,dx \\ &= \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma_\lambda p_j, \sigma_\lambda x) 2(\sum_{j=1}^{k} \xi_{\sigma,j} P w_j \phi_{\varepsilon,j} + O(\xi_\sigma^2))\,dx \\ &= \frac{1}{\pi} \log \frac{1}{\sigma_\lambda} \int\limits_{\mathbb{R}^2} 2\xi_{\sigma,i} w_i \phi_{\varepsilon,i}(1+o(1))\,dx.\end{aligned}$$

Substituting the above equation into the first equation of (6.1), letting $\max\{\sigma, D\} \to 0$, and using the expansion of $\xi_{\sigma,j}$, we arrive at the following nonlocal eigenvalue problem (NLEP):

$$\Delta\phi_j - \phi_j + 2w\phi_j - \frac{2}{1+\tau\lambda} \frac{\int_{\mathbb{R}^2_+} w\phi_j\,dx}{\int_{\mathbb{R}^2_+} w^2\,dx} w^2 = \lambda_0 \phi_j, \;\; j = 1, \cdots, k. \tag{6.5}$$

By Theorem 3.5 in [28], (6.5) has only stable eigenvalues if τ is small enough.

In conclusion, we have shown that the large eigenvalues of the k-peaked solutions given in Theorem 2.1 are all stable if τ is small enough.

7. Study of the small eigenvalues

Now we study the eigenvalue problem (6.1) with respect to small eigenvalues. Namely, we assume that $\lambda_\varepsilon \to 0$ as $\max\{\sigma, D\} \to 0$. We will show that the small eigenvalues in leading order are related to the matrix $\mathcal{M}(\mathbf{p}^0)$ given in (7.3) which is computed from the Green's function. Our main result in this section says that if $\lambda_\varepsilon \to 0$, then in leading order

$$\lambda_\varepsilon \sim \xi_\sigma \sigma_0, \tag{7.1}$$

where σ_0 is an eigenvalue of $\mathcal{M}(\mathbf{p}^0)$. We will show that all the eigenvalues of $\mathcal{M}(\mathbf{p}^0)$ have negative real part provided that the eigenvector is orthogonal to $(1, 1, ..., 1)^T$.

However, for the eigenvector $(1, 1, ..., 1)^T$ the eigenvalue of $\mathcal{M}(\mathbf{p}^0)$ is zero, the leading order term in the eigenvalue expansion vanishes and the next order term is needed to prove stability. To establish it we have to compute the contribution from the boundary curvature. It follows that for a local maximum point of the boundary curvature this eigenvalue has negative real part. Whereas the Green's function part is of order $\epsilon^3 \log \frac{\xi_\sigma}{\epsilon D}$, the part from the boundary curvature is of order ϵ^3. Thus the small eigenvalues of (6.1) are all stable.

To compute the small eigenvalues, we need to expand the spike cluster solution to higher order. Then we expand the eigenfunction and compute the small eigenvalues. This will be done in Appendix B. The key estimates are given in Lemma 9.1.

We compute the small eigenvalues using Lemma 9.1. Comparing l.h.s. and r.h.s., we obtain

$$-\nu_2 \xi_\sigma^2 \mathcal{M}(\mathbf{p}^0)\mathbf{a}_\varepsilon(1+o(1)) = \lambda_\varepsilon \xi_\sigma \mathbf{a}_\varepsilon \int\limits_{\mathbb{R}^2_+} (\frac{\partial w}{\partial y_1})^2\,dy(1+o(1)), \tag{7.2}$$

where ν_2 has been defined in (5.10).

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Further, we have

$$\mathcal{M}(\mathbf{p}^0) = (m_{ij}(\mathbf{p}^0))_{i,j=1}^k, \tag{7.3}$$

where

$$\begin{aligned} m_{ij}(\mathbf{p}) &= \Big[[\nabla^2_{\tau(p_i)} G_0(\sigma p_i, \sigma p_{i-1}) + \nabla^2_{\tau(p_i)} G_0(\sigma p_i, \sigma p_{i+1})]\delta_{ij} \\ &\quad -[\delta_{i,j+1}\nabla^2_{\tau(p_i)} G_0(\sigma p_i, \sigma p_{i-1}) + \delta_{i,j-1}\nabla^2_{\tau(p_i)} G_0(\sigma p_i, \sigma p_{i+1})]\Big]. \end{aligned}$$

Using the estimates for p_i^0 in (5.15) and (5.16), we have

$$\begin{aligned} m_{ij}(\mathbf{p}^0) &= -\frac{h''(p^0)\nu_1}{2\nu_2}\frac{\varepsilon D}{\xi_\sigma}\log\frac{\xi_\sigma}{\varepsilon D}\sigma^2\Big[-(i-1)(k+1-i)\delta_{j,i-1} - i(k-i)\delta_{j,i+1} \\ &\quad +((i-1)(k+1-i)+i(k-i))\delta_{ij}\Big]. \end{aligned}$$

This shows that if all the eigenvalues of $\mathcal{M}(\mathbf{p}^0)$ have positive real part, then the small eigenvalues are stable. On the other hand, if $\mathcal{M}(\mathbf{p}^0)$ has eigenvalues with negative real part, then there are eigenfunctions and eigenvalues to make the system unstable. Next we study the spectrum of the $k \times k$ matrix A defined by

$$\begin{aligned} &a_{s,s} = (s-1)(k-s+1) + s(k-s), \quad s = 1,\cdots,k, \\ &a_{s,s+1} = a_{s+1,s} = -s(k-s), \quad s = 1,\cdots,k, \\ &a_{s,l} = 0, \quad |s-l| > 1. \end{aligned}$$

We have the following result from Lemma 16 in [29]:

Lemma 7.1. *The eigenvalues of the matrix A are given by*

$$\lambda_n = n(n+1), \quad n = 0,\cdots,k-1. \tag{7.4}$$

By Lemma 7.1, the eigenvalues of $\mathcal{M}(\mathbf{p}^0)$ are all positive except for a single eigenvalue zero with eigenvector $(1,1,\ldots,1)^T$.

Equation (7.2) shows that the small eigenvalues λ_ε are

$$\lambda_\varepsilon \sim -\frac{\nu_2\xi_\sigma}{\int_{\mathbb{R}^2_+}(\frac{\partial w}{\partial y_1})^2\,dy}\sigma(\mathcal{M}(\mathbf{p}^0)). \tag{7.5}$$

We remark that the scaling of these small eigenvalues is

$$\lambda_\epsilon \sim c_5\epsilon^3 \log\frac{\xi_\sigma}{\epsilon D}$$

for some $c_5 < 0$.

However, one of the eigenvalues of $\mathcal{M}(\mathbf{p}^0)$ is exactly zero, with eigenvector $(1,1,\ldots,1)^T$. To determine the sign of the real part of this eigenvalue, we have to expand to the next order. By considering contributions for the curvature of the boundary $\partial\Omega$ we have computed in Appendix B that for this eigenvalue

$$\lambda_\epsilon \sim c_6\epsilon^3$$

for some $c_6 < 0$.

To summarise, there are small eigenvalues of two orders which differ by the logarithmic factor $\log\frac{\xi_\sigma}{\epsilon D}$.

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Remark 7.1. A corner can be considered as a point of infinite curvature. Although this case is not included in our analysis, we expect that there will be a stable spike cluster at an outgoing corner which is also a local one-sided maximum for the curvature on both sides of the corner such that a partial spike is located at the corner combined with a finite number of clustered spikes on each side of the corner.

8. Appendix A: linear theory

In this section we prove Proposition 4.1. We follow the Liapunov–Schmidt reduction method. Suppose that to the contrary, there exist sequences $\varepsilon_n, \sigma_n, \mathbf{p}_n$ and Σ_n with $\varepsilon_n, \sigma_n \to 0$ and $\Sigma_n = \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} \in \mathcal{K}^{\perp}_{\varepsilon_n, \mathbf{p}_n}$ such that

$$\begin{cases} \Delta\phi_n - \phi_n + \frac{2U}{V}\phi_n - \frac{U^2}{V^2}\psi_n = f_n, \\ \Delta\psi_n - \sigma^2\psi_n + 2U\phi_n = g_n, \end{cases} \tag{8.1}$$

where

$$\|\pi_{\varepsilon,p} \circ f_n\|_{L^2(\Omega_\varepsilon)} \to 0, \tag{8.2}$$

$$\|\xi_\sigma^{-1} g_n\|_{L^2(\Omega_\varepsilon)} \to 0 \tag{8.3}$$

and

$$\|\phi_n\|_{H^2(\Omega_\varepsilon)} + \|\psi_n\|_{H^2(\Omega_\varepsilon)} = 1. \tag{8.4}$$

We now show that this is impossible. To simplify notation, we omit the index n. In the first step we show that the linearised problem given above tends to a limit problem as $\max\{\sigma, D\} \to 0$.

We define

$$\phi_{\varepsilon,i} = \phi(x)\chi(x - p_i) \text{ for } i = 1, \cdots, k, \tag{8.5}$$

and

$$\phi_{\varepsilon,k+1} = \phi_\varepsilon - \sum_{i=1}^{k} \phi_{\varepsilon,i}. \tag{8.6}$$

It is easy to see that $\phi_{\varepsilon,k+1} = o(1)$ in $H^2(\Omega_\varepsilon)$, since it satisfies

$$\Delta\phi_{\varepsilon,k+1} - \phi_{\varepsilon,k+1} = o(1) \text{ in } H^2(\Omega_\varepsilon).$$

We define $\psi_{\varepsilon,i}$ by

$$\begin{cases} \Delta\psi_{\varepsilon,i} - \sigma^2\psi_{\varepsilon,i} + 2U\phi_{\varepsilon,i} = 0, \text{ in } \Omega_\varepsilon, \\ \frac{\partial\psi_{\varepsilon,i}}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{8.7}$$

Note that since $\xi_\sigma^{-1}\|g_n\|_{L^2(\Omega_\varepsilon)} \to 0$, we also have $\|\psi_{\varepsilon,k+1}\|_{H^2(\Omega_\varepsilon)} = o(1)$.

Next by the equation satisfied by ψ_ε, we have

$$\psi_{\varepsilon,n}(x) = \int_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma x, \sigma y)(2U\phi_\varepsilon - g_n)(y)\,dy. \tag{8.8}$$

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So at $x = p_i$, we calculate

$$
\begin{aligned}
\psi_\varepsilon(p_i) &= \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma p_i, \sigma y)(2U\phi_\varepsilon - g_n)(y)\, dy \\
&= \int\limits_{\Omega_\varepsilon} [\frac{1}{\pi}\log\frac{1}{\sigma|p_i - y|} + \tilde{H}(\sigma p_i, \sigma y)]2U\phi_\varepsilon(y)\, dy \\
&+ \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma p_i, \sigma y) g_n(y)\, dy \\
&= 2\int\limits_{\mathbb{R}^2_+} w(y)\phi_{\varepsilon,i}\, dy(1 + o(1)) + O(\log\frac{1}{\sigma}\|g_n\|_{L^2(\Omega_\varepsilon)}) \\
&= 2\int\limits_{\mathbb{R}^2_+} w(y)\phi_{\varepsilon,i}\, dy + o(1).
\end{aligned}
$$

Substituting the above equation into the first equation of (8.1), letting $\max\{\sigma, D\} \to 0$, we can show that

$$\phi_{\varepsilon,i} \to \phi_i \text{ in } H^2(\mathbb{R}^2_+), \tag{8.9}$$

and

$$\phi_i \in \{\phi \in H^2(\mathbb{R}^2_+) | \int\limits_{\mathbb{R}^2_+} \phi \frac{\partial w}{\partial y_1}\, dy = 0\} := K_0^\perp, \tag{8.10}$$

where ϕ_i is solution of the following nonlocal problem:

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2\int_{\mathbb{R}^2_+} w\phi_i\, dy}{\int_{\mathbb{R}^2_+} w^2\, dy} w^2(y) \in C_0^\perp, \tag{8.11}$$

where $C_0^\perp, K_0^\perp$ denote the orthogonal complements with respect to the scalar product of $L^2(\mathbb{R}^2_+)$ in the space of $H^2(\mathbb{R}^2_+)$ and $L^2(\mathbb{R}^2_+)$ respectively.

By Theorem 1.4 in [18], we know that $\phi_i = 0, i = 1, \cdots, k$.

By taking the limit in the equation satisfied by ψ_ε, we see that this implies that $\psi_\varepsilon \to 0$ in $H^2(\Omega_\varepsilon)$. This contradicts the assumption

$$\|\phi_n\|_{H^2(\Omega_\varepsilon)} + \|\psi_n\|_{H^2(\Omega_\varepsilon)} = 1. \tag{8.12}$$

This proves the boundedness of the linear operator $\mathcal{L}_{\epsilon,\mathbf{p}}$.

To complete the proof of Proposition 4.1, we just need to show that conjugate operator to $\mathcal{L}_{\epsilon,\mathbf{p}}$ (denoted by $\mathcal{L}^*_{\epsilon,\mathbf{p}}$) is injective from $\mathcal{K}^\perp_{\epsilon,}$ to $\mathcal{C}^\perp_{\epsilon,\mathbf{p}}$. The proof for $\mathcal{L}^*_{\epsilon,\mathbf{p}}$ follows almost the same process as for $\mathcal{L}_{\epsilon,\mathbf{p}}$ and therefore it is omitted.

The proof is complete.

9. Appendix B: computation of the small eigenvalues

In this appendix we will compute the small eigenvalues. First we expand the solution to a higher degree of accuracy than in Section 3. Then we expand the eigenfunctions and finally we calculate the small eigenvalues.

9.1. Further expansion of the solution

In this subsection, we further improve our expansion to the solutions derived in Section 3.

First we define

$$u_i(x) = u_\epsilon(x)\chi_\varepsilon(x - p_i),\ i = 1, \cdots, k,$$

where u_ϵ is the exact boundary cluster solution derived in Section 3–5 and χ_ε is the cutoff function given in (3.13). It is easy to see that

$$u(x) = \sum_{i=1}^{k} u_i(x) + O(\varepsilon^5).$$

We will derive an approximation to u_i which is more accurate than that given in Section 3.

Our main idea is to start with a single boundary spike solution of the Gierer–Meinhardt system in a disk B_R of radius R such that the curvature at the centres of the boundary spikes on the disk and the domain $\partial\Omega$ agree. Further, the boundary spike solution in the disk is invariant under rotations of the solution which results in a zero eigenvalue. Then the small eigenvalue of the boundary spike solution in Ω can be computed by a perturbation analysis of the disk case.

The single boundary spike solution (u_0, v_0) in the ball B_R with $\max_{B_R} u_0 = u_0(0, R)$ in polar coordinates solves the following system:

$$\begin{cases} \epsilon^2 \Delta u_0 - u_0 + \frac{u_0^2}{T_R[u_0^2]} = 0 \quad \text{in } B_R, \\ \frac{\partial u_0}{\partial r} = 0 \quad \text{on } \partial B_R, \end{cases} \tag{9.1}$$

where $T_R[u_0^2] = v_0$ is the solution of the inhibitor equation

$$\begin{cases} D\Delta v_0 - v_0 + u_0^2 = 0 \quad \text{in } B_R, \\ \frac{\partial v_0}{\partial r} = 0 \quad \text{on } \partial B_R. \end{cases} \tag{9.2}$$

It can be constructed from the ground state w following the approach in Section 3 by using a fixed-point argument in the space of even functions around $\alpha = 0$ and no Liapunov–Schmidt reduction is needed.

Note that the single-boundary spike solution u_0 is invariant under rotations of the solution. Therefore we can apply $\frac{\partial}{\partial\alpha}$ in (9.1) and get

$$\begin{cases} \epsilon^2 \Delta \frac{\partial u_0}{\partial\alpha} - \frac{\partial u_0}{\partial\alpha} + \frac{2u_0 \frac{\partial u_0}{\partial\alpha}}{T_R[u_0^2]} - \frac{u_0^2}{(T_R[u_0^2])^2} \frac{\partial}{\partial\alpha} T_R[u_0^2] = 0 \quad \text{in } B_R, \\ \frac{\partial^2 u_0}{\partial\alpha\partial r} = 0 \quad \text{on } \partial B_R. \end{cases} \tag{9.3}$$

As a preparation for this perturbation analysis, we represent the boundary $\partial\Omega$ in a neighbourhood of the centre of the spike p_i in polar coordinates and deform it to a circle with the same curvature. This will imply that the perturbation of the boundary will only be in order ϵ^3.

Near the point p_i we have expanded the boundary $\partial\Omega$ in Section 3.1 using Cartesian coordinates. We have derived that $\rho(0) = \rho'(0) = 0$ and $\rho''(0)$ is the curvature of $\partial\Omega$ at p_i. Recall that ρ was used to flatten the boundary $\partial\Omega$ near p_i to a line.

Using polar coordinates (ϕ, r) such that $x_1 - p_1 = r\sin\phi$, $x_2 - p_2 = R - r\sin\phi$, the boundary $\partial\Omega$ can be represented as

$$r = f(\phi).$$

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The radius R will be chosen such that $\rho''(0) = \frac{1}{R}$. We now derive the function $f(\phi)$ from the function $\rho(x_1 - p_1)$ introduced in Section 3.1.

Substituting the Taylor expansion of ρ given in equation (3.8) into $f(\phi)^2$ gives

$$\begin{aligned} f(\phi)^2 &= (x_1 - p_1)^2 + (x_2 - p_2)^2 = r^2 \sin^2 \phi + (R - \rho(r \sin \phi))^2 \\ &= R^2 + r^2 \sin^2 \phi - 2R\rho(r \sin \phi) + \rho^2(r \sin \phi). \end{aligned}$$

We expand

$$f(\phi) = R + \alpha\phi + \beta\phi^2 + \gamma\phi^3 + \delta\phi^4,$$

and compute the coefficients α, β, γ, δ by matching powers ϕ^i for $i = 1, 2, 3, 4$. First we get

$$\alpha = 0.$$

Second we have

$$2\beta = R(1 - \rho''(0)R)$$

which implies that

$$\beta = 0 \text{ provided } \rho''(0) = \frac{1}{R}$$

and from now on we choose R such that this condition is satisfied. Third we compute

$$2R\gamma = -\frac{1}{3}\rho^{(3)}(0)R^4$$

which gives

$$\gamma = -\frac{1}{6}\rho^{(3)}(0)R^3.$$

Finally, we get

$$2R\delta = -\frac{1}{3}R^2 + \frac{1}{3}R^2 - \frac{1}{12}\rho^{(4)}(0)R^5 + \frac{1}{4}\left(\frac{1}{R}\right)^2 R^4$$

which gives

$$\delta = -\frac{1}{24}\rho^{(4)}(0)R^4 + \frac{1}{8}R.$$

To summarise, we have

$$f(\phi) = R + \frac{1}{6}f^{(3)}(0)\phi^3 + \frac{1}{24}f^{(4)}(0)\phi^4 + O(\phi^5),$$

where $f(\phi)$ denotes the radius and ϕ the angle and

$$\begin{aligned} f^{(3)}(0) &= -\rho^{(3)}(0)R^3, \\ f^{(4)}(0) &= -\rho^{(4)}(0)R^4 + 3R. \end{aligned}$$

The point $\phi = 0$ with $f(0) = R$ and $f'(0) = f''(0) = 0$ corresponds to p_i.

Note that $f(0) = R$ and $f'(0) = 0$ follow from $\rho(0) = \rho'(0) = 0$.

Further, we have $f''(0) = 0$ due to the choice of the leading term $f(0) = R$. Since we have $f(\phi) = r$, we get $r = R + O(\phi^3)$ which enables us to replace r by R in the derivation of $f(\phi)$.

In polar coordinates, for $r = R$ we get a point on ∂B_R. We change variables such that for the variables (ϕ, r') at $r' = R$ we get a point on $\partial\Omega$. Thus $r' = R$ has to map into $r = f(\phi)$. This is achieved by defining $r' = r - f(\phi) + R$ for each ϕ.

Then the Laplacian is transformed as follows:

$$\Delta = \Delta' + \frac{1}{r^2}(f'(\phi)^2)\partial_{\phi\phi} - \frac{2}{r^2}f'(\phi)\partial_{r\phi} - \frac{1}{r^2}f''(\phi)\partial_{\phi},$$

using partial derivatives $\partial_r = \frac{\partial}{\partial r}$ etc. and

$$\Delta = \frac{1}{r}\partial_r\left(r\partial_r\right) + \frac{1}{r^2}\partial_{\phi\phi},$$
$$\Delta' = \frac{1}{r'}\partial_{r'}\left(r'\partial_{r'}\right) + \frac{1}{r'^2}\partial_{\phi\phi}$$

for the transformed variables (ϕ, r').

We introduce rescaled variables (α, b) inside the spike such that

$$\epsilon\alpha = \phi, \quad \epsilon b = R - r'.$$

Then in rescaled variables (α, b) we have

$$g(\epsilon\alpha) = R - f(\epsilon\alpha) = \frac{1}{6}\rho^{(3)}(0)R^3\epsilon^3\alpha^3 + \frac{1}{24}\rho^{(4)}(0)R^4\epsilon^4\alpha^4 - \frac{1}{8}R\epsilon^4\alpha^4 + O(\epsilon^5\alpha^5).$$

This implies that in rescaled variables we get

$$\Delta = \Delta' + \frac{1}{r^2}(g'(\epsilon\alpha)^2)\partial_{\alpha\alpha} - \frac{2}{r^2}g'(\epsilon\alpha)\partial_{b\alpha} + \frac{\epsilon}{r^2}g''(\epsilon\alpha)\partial_{\alpha}.$$

Using

$$g'(\epsilon\alpha) = \frac{1}{2}g^{(3)}(0)\epsilon^2\alpha^2 + \frac{1}{6}g^{(4)}(0)\epsilon^3\alpha^3 + O(\epsilon^4\alpha^4)$$

and

$$g''(\epsilon\alpha) = g^{(3)}(0)\epsilon\alpha + \frac{1}{2}g^{(4)}(0)\epsilon^2\alpha^2 + O(\epsilon^3\alpha^3),$$

we have

$$g'(\epsilon\alpha) = O(\epsilon^2\alpha^2)$$

and

$$g''(\epsilon\alpha) = O(\epsilon\alpha).$$

Thus second, third and fourth terms in the Laplacian are small and they can be estimated as follows: $O(\epsilon^4\alpha^4)$, $O(\epsilon^2\alpha^2)$ and $O(\epsilon^2\alpha)$, respectively.

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Comparing with Cartesian coordinates (z_1, z_2) for the ϵ-scale inside the spike used in Section 3.1, by elementary trigonometry we get

$$\epsilon(z_2 - p_2) = R - (R - \epsilon b)\cos(\epsilon\alpha)$$
$$\sim \epsilon b + \epsilon^2 \frac{1}{2} R\alpha^2 - \epsilon^3 \frac{1}{2} b\alpha^2 + O(\epsilon^4)$$

and

$$\epsilon(z_1 - p_1) = (R - \epsilon b)\sin(\epsilon\alpha)$$
$$\sim \epsilon R\alpha - \epsilon^2 \alpha b - \epsilon^3 \frac{1}{6} R\alpha^3 + O(\epsilon^4).$$

Now we approximate the exact solution for the activator u_ϵ as follows:

$$u_\epsilon = \bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 (\bar{v}^{(2)} + \bar{v}^{(3)}) + O(\epsilon^4). \tag{9.4}$$

Near the centre p_i of each spike we have

$$u_i = \xi_{\sigma,i} \bar{w}_i + \epsilon^3 \xi_{\sigma,i} \bar{v}_i^{(1)} + \epsilon^2 \xi_{\sigma,i} (\bar{v}_i^{(2)} + \bar{v}_i^{(3)}) + O(\epsilon^4), \tag{9.5}$$

where we have used the notation

$$\bar{w}_i(x) = \xi_{\sigma,i}^{-1} \bar{w}(x - p_i)\chi_\epsilon(x - p_i), \quad \bar{v}_i^{(j)}(x) = \xi_{\sigma,i}^{-1} \bar{v}^{(j)}(x - p_i)\chi_\varepsilon(x - p_i) \tag{9.6}$$

for $i = 1, 2, \ldots, k$, $j = 1, 2, 3$ and χ_ε is the cutoff function defined in (3.13).

Let us derive the terms in this expansion step by step. We start from the single boundary spike solution u_0 defined in (9.1) in a ball of radius R such that $\rho''(0) = \frac{1}{R}$. Using polar coordinates, we can represent $u_0(\epsilon\alpha, R - r)$ and this function satisfies the Neumann boundary condition in a ball:

$$\left.\frac{\partial u_0}{\partial r}\right|_{r=R} = 0.$$

However, the function u_0 does not satisfy the boundary condition at $\partial\Omega$ (and for $r = R$ we reach the boundary of the disk (circle) but not the domain boundary $\partial\Omega$). Recall that $r - f(\phi) = r' - R$. Thus, to get a function with a better approximation to the boundary condition at $\partial\Omega$ (and such that for $r = R$ we reach $\partial\Omega$), we define

$$\bar{w}(\epsilon\alpha, r') = u_0(\epsilon\alpha, r).$$

Then for $r' = R$, we have

$$\bar{w}(\epsilon\alpha, R) = u_0(\epsilon\alpha, f(\epsilon\alpha)),$$

i.e. for $r' = R$ the arguments of the function $\bar{w}$ are contained in $\partial\Omega$ and the arguments of u_0 are contained in ∂B_R. This means that the function f deforms the boundary to a circle (in the same way as in Section 3.1 ρ deforms the boundary to a straight line). Since the circle also takes into account the curvature it gives a better approximation to the boundary than the straight line and the approximate spike solution will give a better approximation to the exact solution than the approximations in Section 3.1. Note that the boundary spike solution in the ball is invariant under rotation (in the same way as the boundary spike is translation invariant in half space).

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Now we calculate the radial derivative of $\bar{w}$ as follows:

$$\frac{\partial \bar{w}(\epsilon\alpha, r')}{\partial r'}|_{r'=R} = \frac{\partial u_0(\epsilon\alpha, r)}{\partial r}|_{r=f(r)} = 0. \tag{9.7}$$

The outward unit normal vector in polar coordinates is given by

$$\nu = \frac{1}{\sqrt{f'(\phi)^2 + f(\phi)^2}}(-f'(\phi), f(\phi)).$$

Using rescaled variables, this implies

$$\begin{aligned}\nu &= \frac{1}{\sqrt{g'(\epsilon\alpha)^2 + (R - g(\epsilon\alpha))^2}}(g'(\epsilon\alpha), R - g(\epsilon\alpha)) \\ &= \frac{1}{\sqrt{\frac{1}{4}g^{(3)}(0)^2\epsilon^4\alpha^4 + (R - \frac{1}{6}g^{(3)}(0)\epsilon^3\alpha^3)^2 + O(\epsilon^4\alpha^4)}} \\ &\times(\frac{1}{2}g^{(3)}(0)\epsilon^2\alpha^2 + \frac{1}{6}g^{(4)}(0)\epsilon^3\alpha^3 + O(\epsilon^4\alpha^4), R - \frac{1}{6}f^{(3)}(0)\epsilon^3\alpha^3)\end{aligned}$$

Subtracting the radial unit vector $e_r = (0, 1)$, this implies

$$\nu - e_r = \frac{(\frac{1}{2}g^{(3)}(0)\epsilon^2\alpha^2 + \frac{1}{6}g^{(4)}(0)\epsilon^3\alpha^3, O(\epsilon^3\alpha^3))}{R + O(\epsilon^3\alpha^3)}.$$

Using that $\|\bar{w}_i - w\|_{H^2(\Omega_\epsilon)} = O(\epsilon\alpha)$ and that $\bar{w}$ satisfies (9.7), the outward normal derivative of $\bar{w}$ is computed as

$$\begin{cases} \frac{\partial \bar{w}_i}{\partial \nu} = \left(\frac{\partial \bar{w}_i}{\partial \nu} - \frac{\partial \bar{w}_i}{\partial r}\right) + \frac{\partial \bar{w}_i}{\partial r} \\ = \frac{1}{R}w'(|y|)(\frac{1}{2}g^{(3)}(0)\epsilon^2\alpha^2 + \frac{1}{6}g^{(4)}(0)\epsilon^3\alpha^3 + O(\epsilon^4\alpha^4)). \end{cases} \tag{9.8}$$

Now we compute the terms $\epsilon^3\bar{v}^{(1)}$ (even around $\alpha = 0$) and $\epsilon^2\bar{v}^{(2)}$ (odd around $\alpha = 0$) in the expansion (9.5) near p_i such that the solution satisfies the Neumann boundary condition to higher order.

Let $\bar{v}_i^{(1)}$ satisfy

$$\begin{cases} \Delta v - v = 0 \quad \text{in } B_R, \\ \frac{\partial v}{\partial b} = \frac{1}{6R}g^{(4)}(0)\alpha^3 w'(|y|) \text{ on } \partial B_R. \end{cases}$$

Let $\bar{v}_i^{(2)}$ be given by

$$\begin{cases} \Delta v - v = 0 \quad \text{in } B_R, \\ \frac{\partial v}{\partial b} = \frac{1}{2R}g^{(3)}(0)\alpha^2 w'(|y|) \text{ on } \partial B_R. \end{cases}$$

Substituting the expansion (9.4) of the solution in the activator equation, we get

$$\begin{aligned}& S_1(\bar{w} + \epsilon^3\bar{v}^{(1)} + \epsilon^2\bar{v}^{(2)} + O(\epsilon^4), T[\bar{w} + \epsilon^3\bar{v}^{(1)} + \epsilon^2\bar{v}^{(2)} + O(\epsilon^4)]) \\ &= \sum_{i=1}^{k}\xi_{\sigma,i}((\bar{w}_i + \epsilon^3\bar{v}_i^{(1)} + \epsilon^2\bar{v}_i^{(2)} + O(\epsilon^4))^2 - \bar{w}_i^2) + \sum_{i=1}^{k}\xi_{\sigma,i}^2\bar{w}_i^2(\frac{1}{V(x)} - \frac{1}{V(p_i)}) + O(\xi_\sigma\varepsilon^4).\end{aligned}$$

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We calculate for $x = p_i + z$

$$[(\bar{w}_i(x) + \epsilon^3 \bar{v}^{(1)}(x) + \epsilon^2 \bar{v}^{(2)}(x) + O(\epsilon^4)]^2 - \bar{w}_i^2(x) = 2\bar{w}_i(z)(\varepsilon^3 \bar{v}_i^{(1)}(z) + \varepsilon^2 \bar{v}_i^{(2)}(z) + O(\varepsilon^4)) \\ := \varepsilon^3 \bar{R}_{1,i}(z) + \varepsilon^2 \bar{R}_{2,i}(z) + O(\varepsilon^4),$$

where $\bar{R}_{1,i}(z) = 2\bar{w}_i(z) v_i^{(1)}(z)$, $\bar{R}_{2,i} = 2\bar{w}_i(z) v_i^{(2)}(z)$. Further, we recall that by (4.4) we have

$$\frac{1}{V(p_i+z)} - \frac{1}{V(p_i)} = \frac{1}{V(p_i)^2}(-\xi_{\sigma,i}^2 R_1(z) - \xi_{\sigma,i}^2 R_2(z) + h.o.t). \tag{9.9}$$

By the reduced problem in Section 5, we have

$$\xi_{\sigma,i} \bar{R}_{2,i} - \xi_{\sigma,i}^2 \bar{w}_i^2 R_2 + O(\xi_\sigma \epsilon^3) \perp \frac{\partial \bar{w}_i}{\partial \alpha}.$$

Therefore we can add another contribution $\bar{v}_i^{(3)}$ to the solution such that $\varepsilon^2 \bar{v}_i^{(3)}$ satisfies

$$\begin{cases} \tilde{L}^{(i)} v = \epsilon^2 \bar{R}_{2,i} - \xi_{\sigma,i} \bar{w}_i^2 R_2 + O(\xi_\sigma \epsilon^3) & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{L}^{(i)} \phi = \Delta \phi - \phi + \frac{2\xi_{\sigma,i}(\bar{w}_i + \epsilon^2 \bar{v}_i^{(2)})\phi}{T\left[\xi_{\sigma,i}^2 (\bar{w}_i + \epsilon^2 \bar{v}_i^{(2)})^2\right]} \\ - \frac{\xi_{\sigma,i}^2 (\bar{w}_i + \epsilon^2 \bar{v}_i^{(2)})^2}{T\left[\xi_{\sigma,i}^2 (\bar{w}_i + \epsilon^2 \bar{v}_i^{(2)})^2\right]^2} T\left[2\xi_{\sigma,i}(\bar{w}_i + \epsilon^2 \bar{v}_i^{(2)})\phi\right].$$

Setting

$$\bar{v}^{(3)} = \sum_{i=1}^k \xi_{\sigma,i} \bar{v}_i^{(3)}$$

and adding this part to the solution will cancel out the odd terms (with respect to $\alpha = 0$) of order ϵ^2 in the activator equation and we get

$$S_1(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2(\bar{v}^{(2)} + \bar{v}^{(3)}) + O(\epsilon^4), T[\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2(\bar{v}^{(2)} + \bar{v}^{(3)}) + O(\epsilon^4)]) \\ = \sum_{i=1}^k 2\xi_{\sigma,i} \epsilon^3 \bar{w}_i \bar{v}_i^{(1)} + \sum_{i=1}^k \xi_{\sigma,i}^2 \bar{w}_i^2 (-\frac{1}{(V(p_i))^2} \frac{1}{2} V''(p_i)(x-p_i)^2) + O(\epsilon^4).$$

Taking the derivative $\frac{\partial}{\partial \alpha}$ in this relation near $x = p_i$, we compute

$$\frac{\partial}{\partial \alpha} S_1(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2(\bar{v}^{(2)} + \bar{v}^{(3)}) + O(\epsilon^4), T[\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2(\bar{v}^{(2)} + \bar{v}^{(3)}) + O(\epsilon^4)]) \\ = 2\sum_{i=1}^k \xi_{\sigma,i} \frac{\partial \bar{w}_i(z)}{\partial \alpha}(\varepsilon^3 \bar{v}_i^{(1)}(z) + \varepsilon^2(\bar{v}_i^{(2)}(z) + \bar{v}^{(3)}(z)))$$

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$$+2\sum_{i=1}^{k}\xi_{\sigma,i}\bar{w}_i(z)\frac{\partial}{\partial\alpha}(\varepsilon^3\bar{v}_i^{(1)}(z)+\varepsilon^2(\bar{v}_i^{(2)}(z)+\bar{v}^{(3)}(z)))$$

$$+\sum_{i=1}^{k}\xi_{\sigma,i}^2 2\bar{w}_i\frac{\partial\bar{w}_i}{\partial\alpha}(\frac{1}{V(x)}-\frac{1}{V(p_i)})$$

$$+\sum_{i=1}^{k}\xi_{\sigma,i}^2\bar{w}_i^2\frac{\partial}{\partial\alpha}(\frac{1}{V(x)}-\frac{1}{V(p_i)})+O(\varepsilon^4)$$

$$=\sum_{i=1}^{k}2\xi_{\sigma,i}\epsilon^3\left(\frac{\partial\bar{w}_i}{\partial\alpha}\bar{v}_i^{(1)}+\bar{w}_i\frac{\partial\bar{v}_i^{(1)}}{\partial\alpha}\right)$$

$$+\sum_{i=1}^{k}\xi_{\sigma,i}^2\bar{w}_i^2(-\frac{1}{(V(p_i))^2}\frac{1}{2}V''(p_i)(x-p_i)^2)+O(\varepsilon^4).$$

On the other hand,

$$\frac{\partial}{\partial\alpha}S_1(\bar{w}+\epsilon^3\bar{v}^{(1)}+\epsilon^2(\bar{v}^{(2)}+\bar{v}^{(3)})+O(\epsilon^4),T[\bar{w}+\epsilon^3\bar{v}^{(1)}+\epsilon^2(\bar{v}^{(2)}+\bar{v}^{(3)})+O(\epsilon^4)])$$

$$=L^{(i)}\left[\xi_{\sigma,i}(\frac{\partial\bar{w}_i}{\partial\alpha}+\epsilon^3\frac{\partial\bar{v}_i^{(1)}}{\partial\alpha}+\epsilon^2\frac{\partial\bar{v}_i^{(2)}}{\partial\alpha}+\epsilon^2\frac{\partial\bar{v}^{(3)}}{\partial\alpha})\right]+O(\epsilon^4), \tag{9.10}$$

where

$$L^{(i)}\phi=\Delta\phi-\phi+\frac{2\xi_{\sigma,i}(\bar{w}_i+\epsilon^3\bar{v}_i^{(1)}+\epsilon^2\bar{v}_i^{(2)}+\epsilon^2\bar{v}_i^{(3)})\phi}{T\left[\xi_{\sigma,i}^2(\bar{w}_i+\epsilon^3\bar{v}_i^{(1)}+\epsilon^2\bar{v}_i^{(2)}+\epsilon^2\bar{v}_i^{(3)})^2\right]}$$

$$-\frac{\xi_{\sigma,i}^2(\bar{w}_i+\epsilon^3\bar{v}_i^{(1)}+\epsilon^2\bar{v}_i^{(2)}+\epsilon^2\bar{v}_i^{(3)})^2}{T\left[(\bar{w}_i+\epsilon^3\bar{v}_i^{(1)}+\epsilon^2\bar{v}_i^{(2)}+\epsilon^2\bar{v}_i^{(3)})^2\right]^2}T\left[2\xi_{\sigma,i}(\bar{w}_i+\epsilon^3\bar{v}_i^{(1)}+\epsilon^2\bar{v}_i^{(2)}+\epsilon^2\bar{v}_i^{(3)})\phi\right].$$

Finally, by an expansion similar to (9.8) we have

$$\frac{\partial\bar{v}_i^{(3)}}{\partial\nu}=O(\epsilon^3)\quad\text{ on }\partial\Omega$$

and

$$\frac{\partial}{\partial\nu}\frac{\partial\bar{v}_i^{(3)}}{\partial\alpha}=O(\epsilon^3)\quad\text{ on }\partial\Omega.$$

9.2. Expansion of the eigenfunction

We define the approximate kernels to be

$$\mathcal{K}_{\varepsilon,\mathbf{p}}:=\text{Span}\{\frac{\partial u_i}{\partial\tau(p_i)},i=1,\cdots,k\},$$

$$\mathcal{C}_{\varepsilon,\mathbf{p}}:=\text{Span}\{\frac{\partial u_i}{\partial\tau(p_i)},i=1,\cdots,k\}.$$

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Then we expand the eigenfunction as follows:

$$\phi_\epsilon = \sum_{i=1}^{k} a_{i,\varepsilon}\left(\frac{\partial \bar{w}_i}{\partial \alpha} + \epsilon^3 \bar{v}^{(1)}_{eig,i} + \epsilon^2 \bar{v}^{(2)}_{eig,i} + \epsilon^2 \frac{\partial \bar{v}^{(3)}_i}{\partial \alpha}\right) + \phi^\perp_\epsilon + O(\epsilon^4),$$

$$:= \sum_{i=1}^{k} a_{\varepsilon,i}\phi_{i,\varepsilon} + \phi^\perp_\epsilon + O(\epsilon^4), \tag{9.11}$$

where $\phi^\perp_\epsilon \in \mathcal{K}^\perp_{\varepsilon,\mathbf{p}}$.

Suppose that $\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1$. Then $|a_{j,\varepsilon}| \leq C$. Let us put

$$\mathbf{a}^\varepsilon := (a_{1,\varepsilon}, \cdots, a_{k,\varepsilon})^T \tag{9.12}$$

Then for a subsequence and

$$a_{i,0} = \lim_{\varepsilon\to 0} a_{i,\varepsilon},\ \mathbf{a}^0 := (a_{1,0}, \cdots, a_{k,0}). \tag{9.13}$$

The decomposition of ϕ_ε in (9.11) implies that

$$\psi_\varepsilon = \sum_{i=1}^{k} a_{i,\varepsilon}\psi_{i,\varepsilon} + \psi^\perp_\varepsilon, \tag{9.14}$$

where $\psi_{i,\varepsilon}$ satisfies

$$\begin{cases} \Delta\psi_{i,\varepsilon} - \sigma^2(1+\tau\lambda_\varepsilon)\psi_{i,\varepsilon} + 2u\phi_{i,\varepsilon} = 0 \text{ in } \Omega_\varepsilon \\ \frac{\partial \psi_{i,\varepsilon}}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{9.15}$$

which we also write as $\psi_{i,\varepsilon} = T_{\tau\lambda_\varepsilon}[\phi_{i,\varepsilon}]$, and $\psi^\perp_\varepsilon$ is given by

$$\begin{cases} \Delta\psi^\perp_\varepsilon - \sigma^2(1+\tau\lambda_\varepsilon)\psi^\perp_\varepsilon + 2u\phi^\perp_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ \frac{\partial \psi^\perp_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{9.16}$$

which we also represent as $\psi^\perp_\varepsilon = T_{\tau\lambda_\varepsilon}[\phi^\perp_\varepsilon]$.

Let us first consider the leading term of ϕ_ε. For $\frac{\partial \bar{w}_i}{\partial \alpha}$ we get, using (9.8),

$$\begin{cases} \Delta \frac{\partial \bar{w}_i}{\partial \alpha} - \frac{\partial \bar{w}_i}{\partial \alpha} + \frac{2\bar{w}_i \frac{\partial \bar{w}_i}{\partial \alpha}}{T[\bar{w}_i^2]} - \frac{\bar{w}_i^2}{(T[\bar{w}_i^2])^2}\frac{\partial}{\partial \alpha}T[\bar{w}_i^2] = O(\epsilon^4\alpha^3) \quad \text{ in } \Omega, \\ \frac{\partial}{\partial b}\frac{\partial \bar{w}_i}{\partial \alpha} = \frac{1}{R}\frac{\partial w'(|y|)}{\partial \alpha}(\frac{1}{2}g^{(3)}(0)\epsilon^2\alpha^2 + \frac{1}{6}g^{(4)}(0)\epsilon^3\alpha^3)(1+O(\epsilon\alpha)) \quad \text{ on } \partial\Omega. \end{cases}$$

Therefore, expanding the boundary condition as we did above for $\bar{w}_i$, we define $\bar{v}^{(1)}_{eig,i}$ as the unique solution of

$$\begin{cases} \Delta v - v = 0 \quad \text{ in } B_R, \\ \frac{\partial v}{\partial b} = \frac{1}{6R}\frac{\partial w'(|y|)}{\partial \alpha} g^{(4)}(0)\epsilon^3\alpha^3 \quad \text{ on } \partial B_R. \end{cases}$$

Similarly, let $\bar{v}^{(2)}_{eig,i}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0 & \text{in } B_R, \\ \frac{\partial v}{\partial b} = \frac{1}{2R} \frac{\partial w'(|y|)}{\partial \alpha} g^{(3)}(0) \epsilon^2 \alpha^2 & \text{on } \partial B_R. \end{cases}$$

Let us compare this with the derivative $\frac{\partial \bar{v}^{(1)}}{\partial \alpha}$ which satisfies

$$\begin{cases} \Delta v - v = 0 \\ \frac{\partial v}{\partial b} = \frac{\partial}{\partial \alpha} [\frac{1}{6R} w'(|y|) g^{(4)}(0) \epsilon^3 \alpha^3] & \text{on } \partial B_R. \end{cases}$$

Further, $\frac{\partial \bar{v}^{(2)}}{\partial \alpha}$ solves

$$\begin{cases} \Delta v - v = 0 & \text{in } B_R, \\ \frac{\partial v}{\partial b} = \frac{\partial}{\partial \alpha} [\frac{1}{2R} w'(|y|) g^{(3)}(0) \epsilon^2 \alpha^2] & \text{on } \partial B_R. \end{cases}$$

Using (9.10), we get

$$\begin{aligned} & L\left[\sum_{i=1}^k a_{i,\varepsilon}\left(\frac{\partial \bar{w}_i}{\partial \alpha} + \epsilon^3 \bar{v}^{(1)}_{eig,i} + \epsilon^2 \bar{v}^{(2)}_{eig,i} + \epsilon^2 \frac{\partial \bar{v}^{(3)}_i}{\partial \alpha}\right)\right] \\ & -L\left[\sum_{i=1}^k a_{i,\varepsilon}\left(\frac{\partial \bar{w}_i}{\partial \alpha} + \epsilon^3 \frac{\partial \bar{v}^{(1)}_i}{\partial \alpha} + \epsilon^2 \frac{\partial \bar{v}^{(2)}_i}{\partial \alpha} + \epsilon^2 \frac{\partial \bar{v}^{(3)}_i}{\partial \alpha}\right)\right] \\ & = 2\sum_{i=1}^k a_{i,\varepsilon} \epsilon^3 \bar{w}_i \bar{v}^{(1)}_{R,i} \\ & + \sum_{i=1}^k a_{i,\varepsilon} \left(\frac{\bar{w}_i^2}{v^2} \frac{\partial v}{\partial \tau(p_i)} - \frac{1}{\xi_{\sigma,i}} \frac{u^2}{v^2} \psi_{i,\varepsilon}\right) + O(\varepsilon^4), \end{aligned}$$

where

$$\begin{aligned} L\phi = \Delta\phi - \phi + \frac{2(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 \bar{v}^{(2)} + \epsilon^2 \bar{v}^{(3)})\phi}{T\left[(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 \bar{v}^{(2)} + \epsilon^2 \bar{v}^{(3)})^2\right]} \\ - \frac{(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 \bar{v}^{(2)} + \epsilon^2 \bar{v}^{(3)})^2}{T\left[(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 \bar{v}^{(2)} + \epsilon^2 \bar{v}^{(3)})^2\right]^2} T\left[2\xi_{\sigma,i}(\bar{w} + \epsilon^3 \bar{v}^{(1)} + \epsilon^2 \bar{v}^{(2)} + \epsilon^2 \bar{v}^{(3)})\phi\right] \end{aligned}$$

and the remainder $\bar{v}^{(1)}_{R,i}$ is given by the difference of the previous two contributions as follows: $\bar{v}^{(1)}_{R,i} = \bar{v}^{(1)}_{eig,i} - \frac{\partial \bar{v}^{(1)}}{\partial \alpha}$ which satisfies

$$\begin{cases} \Delta v - v = 0 & \text{in } B_R, \\ \frac{\partial v}{\partial b} = -\frac{1}{6R} w'(|y|) g^{(4)}(0) 3\alpha^2 (1 + O(\epsilon\alpha)) & \text{on } \partial B_R. \end{cases} \tag{9.17}$$

We note that $\bar{v}^{(1)}_{R,i}$ is an even function around $\alpha = 0$.

Substituting the decompositions of ϕ_ε and ψ_ε into (6.1), we have

$$\begin{aligned} & L\left[\sum_{i=1}^k a_{i,\varepsilon}\left(\frac{\partial \bar{w}_i}{\partial \alpha} + \epsilon^3 \bar{v}^{(1)}_{eig,i} + \epsilon^2 \bar{v}^{(2)}_{eig,i} + \epsilon^2 \frac{\partial \bar{v}^{(3)}_i}{\partial \alpha}\right) + \phi^\perp_\epsilon\right] \\ & = \sum_{i=1}^k 2\epsilon^3 \bar{w}_i \bar{v}^{(1)}_{R,i} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\xi_\sigma}\sum_{i=1}^k a_{i,\varepsilon}(\frac{u_i^2}{v^2}\frac{\partial v}{\partial\tau(p_i)}-\frac{u^2}{v^2}\psi_{i,\varepsilon})\\
&+\Delta\phi_\varepsilon^\perp-\phi_\varepsilon^\perp+\frac{2u}{v}\phi_\varepsilon^\perp-\frac{u^2}{v^2}\psi_\varepsilon^\perp-\lambda_\varepsilon\phi_\varepsilon^\perp\\
&=\frac{1}{\xi_\sigma}\lambda_\varepsilon\sum_{i=1}^k a_{i,\varepsilon}\frac{\partial u_i}{\partial\tau(p_i)}+O(\varepsilon^4).
\end{aligned}
\tag{9.18}
$$

Set

$$
I_3=\sum_{i=1}^k 2\epsilon^3\bar{w}_i\bar{v}_{R,i}^{(1)},
$$

$$
I_4=\frac{1}{\xi_\sigma}\sum_{i=1}^k a_{i,\varepsilon}(\frac{u_i^2}{v^2}\frac{\partial v}{\partial\tau(p_i)}-\frac{u^2}{v^2}\psi_{i,\varepsilon})
$$

and

$$
I_5=\Delta\phi_\varepsilon^\perp-\phi_\varepsilon^\perp+\frac{2u}{v}\phi_\varepsilon^\perp-\frac{u^2}{v^2}\psi_\varepsilon^\perp-\lambda\phi_\varepsilon^\perp.
$$

We first compute

$$
\begin{aligned}
I_4 &=\frac{1}{\xi_\sigma}\sum_{i=1}^k a_{i,\varepsilon}(\frac{u_i^2}{v^2}\frac{\partial v}{\partial\tau(p_i)}-\frac{u^2}{v^2}\psi_{i,\varepsilon})+O(\varepsilon^4)\\
&=\frac{1}{\xi_\sigma}\sum_{i=1}^k a_{i,\varepsilon}(\frac{u_i^2}{v^2}\frac{\partial v}{\partial\tau(p_i)}-\frac{u_i^2}{v^2}\psi_{i,\varepsilon})-\frac{1}{\xi_\sigma}\sum_{i=1}^k\sum_{j\neq i}a_{i,\varepsilon}\frac{u_j^2}{v^2}\psi_{i,\varepsilon}+O(\varepsilon^4).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{1}{\xi_\sigma}\sum_{i=1}^k\sum_{|i-j|\geq 2}a_{i,\varepsilon}\frac{u_j^2}{v^2}\psi_{i,\varepsilon}\\
&=\xi_\sigma\sum_{i=1}^k\sum_{|i-j|\geq 2}|\nabla_{p_i}G_{\sqrt{D}}(\sigma_\lambda p_i,\sigma_\lambda p_j)|(1+o(1))\\
&=O(\xi_\sigma\sigma\log\varepsilon(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2)\\
&=O(\sigma(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2),
\end{aligned}
$$

we can estimate I_4 as follows:

$$
\begin{aligned}
I_4 &=\frac{1}{\xi_\sigma}\sum_{i=1}^k\sum_{j=1}^k a_{i,\varepsilon}\frac{u_j^2}{v^2}(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon}))+O(\varepsilon^4)\\
&=-\frac{1}{\xi_\sigma}\sum_{i=1}^k\sum_{|i-j|=1}a_{i,\varepsilon}\frac{u_j^2}{v^2}\psi_{i,\varepsilon}+O(\varepsilon^4)+O(\sigma^2\lambda_\varepsilon)+O(\sigma(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2),
\end{aligned}
$$

where we use the equation satisfied by $\psi_{i,\varepsilon}$.

9.3. Expansion of the small eigenvalues

Multiplying both sides of (9.18) by $\frac{\partial u_l}{\partial \tau(p_l)}$ and integrating over Ω_ε, we have

$$\begin{aligned} r.h.s &= \lambda_\varepsilon \frac{1}{\xi_\sigma} \sum_{i=1}^k a_{i,\varepsilon} \int_{\Omega_\varepsilon} \frac{\partial u_i}{\partial \tau(p_i)} \frac{\partial u_l}{\partial \tau(p_l)}\, dx \\ &= \lambda_\varepsilon \xi_\sigma a_{l,\varepsilon} \int_{\mathbb{R}^2_+} (\frac{\partial w}{\partial y_1})^2\, dy(1+o(1)). \end{aligned}$$

Using the estimate I_4, we have

$$\begin{aligned} l.h.s &= \int_{\Omega_\varepsilon} (\Delta\phi_\varepsilon^\perp - \phi_\varepsilon^\perp + \frac{2u}{v}\phi_\varepsilon^\perp - \frac{u^2}{v^2}\psi_\varepsilon^\perp - \lambda_\epsilon \phi_\varepsilon^\perp) \frac{\partial u_l}{\partial \tau(p_l)}\, dx \\ &+ \int_{\Omega_\varepsilon} \frac{1}{\xi_\sigma} \sum_{i=1}^k a_{i,\varepsilon} (\frac{u_i^2}{v^2} \frac{\partial v}{\partial \tau(p_i)} - \frac{u^2}{v^2}\psi_{i,\varepsilon}) \frac{\partial u_l}{\partial \tau(p_l)}\, dx + O(\varepsilon^4) \\ &= \int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \frac{\partial v}{\partial \tau(p_l)} \phi_\varepsilon^\perp\, dx - \int_{\Omega_\varepsilon} \frac{u^2}{v^2}\psi_\varepsilon^\perp \frac{\partial u_l}{\partial \tau(p_l)}\, dx - \lambda_\varepsilon \int_{\Omega_\varepsilon} \phi_\varepsilon^\perp \frac{\partial u_l}{\partial \tau(p_l)}\, dx \\ &+ \int_{\Omega_\varepsilon} \frac{1}{\xi_\sigma} \sum_{i=1}^k \sum_{j=1}^k a_{i,\varepsilon} \frac{u_j^2}{v^2} (\frac{\partial v}{\partial \tau(p_j)}\delta_{ij} - \psi_{i,\varepsilon})) \frac{\partial u_l}{\partial \tau(p_l)}\, dx \\ &+ \sum_{i=1}^k \int_{\Omega_\varepsilon} 2a_{i,\varepsilon}\varepsilon^3 \bar{w}_i \bar{v}_{R,i}^{(1)} \frac{\partial u_l}{\partial \tau(p_l)} + O(\varepsilon^4) \\ &= J_{1,l} + J_{2,l} + J_{3,l} + J_{4,l} + J_{5,l} + O(\varepsilon^4), \end{aligned}$$

where $J_{i,l}$ are defined as the integrals in the last equality. We divide our proof into several steps.

The following lemma contains the key estimates:

Lemma 9.1. *We have*

$$J_{1,l} = o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \tag{9.19}$$

$$J_{2,l} = o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \tag{9.20}$$

$$J_{3,l} = o(\xi_\sigma \lambda_\varepsilon), \tag{9.21}$$

$$\begin{aligned} J_{4,l} = (\int_{\mathbb{R}^2_+} w^2\, dy \int_{\mathbb{R}^2_+} w^2 \frac{\partial w}{\partial y_1} y_1\, dy) \xi_\sigma^2 (1+o(1)) \times \Big[[\sum_{|i-l|=1} \nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_i, \sigma p_l)] a_{l,\varepsilon} \\ - [a_{l-1,\varepsilon} \nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_{l-1}) + a_{l+1,\varepsilon} \nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_{l+1})] \Big] + o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}) \end{aligned} \tag{9.22}$$

$$J_{5,l} = \epsilon^3 \xi_{\sigma,l} a_{l,\varepsilon} \frac{3}{4} \nu_1 \frac{\partial^2}{\partial \tau(\varepsilon p_l)^2} h(\varepsilon p_l) + o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \tag{9.23}$$

where $a_{i,\varepsilon}$ has been defined in (9.11).

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Proof. We first study the asymptotic behaviour of $\psi_{j,\varepsilon}$.

Note that for $l \neq k$, we have

$$\begin{aligned}\psi_{k,\varepsilon}(p_l) &= \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma_\lambda p_l, \sigma_\lambda y) 2u\phi_{k,\varepsilon}(y)\, dy \\ &= \nabla_{\tau(p_k)} G_{\sqrt{D}}(\sigma_\lambda p_l, \sigma_\lambda p_k) \int\limits_{\mathbb{R}^2_+} 2\xi^2_{\sigma,k} w \frac{\partial w}{\partial y_1} y\, dy + O(\xi^2_\sigma \sigma^2 \frac{\varepsilon D}{\xi_\sigma}) \\ &= \nabla_{\tau(p_k)} G_{\sqrt{D}}(\sigma p_l, \sigma p_k) \int\limits_{\mathbb{R}^2_+} 2\xi^2_{\sigma,k} w \frac{\partial w}{\partial y_1} y\, dy (1 + O(\lambda_\varepsilon \log \varepsilon)) + o(\xi^2_\sigma \sigma^2 \frac{\varepsilon D}{\xi_\sigma}).\end{aligned} \tag{9.24}$$

Next we compute $\psi_{l,\varepsilon} - \frac{\partial v}{\partial \tau(p_l)}$ near p_l:

$$\begin{aligned} v(x) &= \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma x, \sigma y) u^2(y)\, dy \\ &= \int\limits_{\Omega_\varepsilon} \frac{1}{\pi} \log \frac{1}{\sigma|x-y|} u_l^2(y)\, dy + \int\limits_{\Omega_\varepsilon} \tilde{H}(\sigma x, \sigma y) u_l^2(y)\, dy \\ &\quad + \sum_{i \neq l} \int\limits_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma x, \sigma y) u_i^2(y)\, dy + O(\varepsilon^4), \end{aligned}$$

so

$$\begin{aligned} \frac{\partial v}{\partial \tau(p_l)}(x) &= \sum_{|i-l|=1} \nabla_{\tau(p_i)} G_{\sqrt{D}}(\sigma x, \sigma p_i) \xi^2_{\sigma,i} \int\limits_{\mathbb{R}^2_+} w^2\, dy + O(\xi^2_\sigma \sigma |\log \varepsilon| (\frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma})^2) \\ &= \sum_{|i-l|=1} \nabla_{\tau(p_i)} G_{\sqrt{D}}(\sigma x, \sigma p_l) \xi^2_{\sigma,i} \int\limits_{\mathbb{R}^2_+} w^2\, dy + o(\xi^2_\sigma \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \end{aligned}$$

where we have used the relation $e^{-\frac{1}{\sqrt{D}}} \ll \varepsilon$.

Thus

$$\begin{aligned} \frac{\partial v}{\partial \tau(p_l)} - \psi_{l,\varepsilon}(p_l) &= \sum_{|i-l|=1} \nabla_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_i) \xi^2_{\sigma,i} \int\limits_{\mathbb{R}^2_+} w^2\, dy \\ &\quad + o(\xi^2_\sigma \sigma^2 \frac{\varepsilon D}{\xi_\sigma}). \end{aligned} \tag{9.25}$$

Combining (9.24) and (9.25), we have

$$\begin{aligned} (\psi_{i,\varepsilon} - \frac{\partial v}{\partial \tau(p_i)} \delta_{il})(p_l) &= \nabla_{\tau(p_i)} G_{\sqrt{D}}(\sigma p_l, \sigma p_i) (2\xi^2_{\sigma,i} \int\limits_{\mathbb{R}^2_+} w \frac{\partial w}{\partial y_1} y_1\, dy)[\delta_{i,l-1} + \delta_{i,l+1}] \\ &\quad - \delta_{il} \sum_{|m-l|=1} \nabla_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_m)(\xi^2_{\sigma,m} \int\limits_{\mathbb{R}^2_+} w^2\, dy) \\ &\quad + o(\xi^2_\sigma \sigma^2 \frac{\varepsilon D}{\xi_\sigma}) + o(\xi^2_\sigma \lambda_\varepsilon). \end{aligned} \tag{9.26}$$

Similarly, we have the following:

$$\begin{aligned}
&\psi_{i,\varepsilon}(p_l+(y_1,0))-\psi_{i,\varepsilon}(p_l) \qquad (9.27)\\
&=\int_{\Omega_\varepsilon}(G_{\sqrt{D}}(\sigma_\lambda(p_l+y),\sigma_\lambda x)-G_{\sqrt{D}}(\sigma_\lambda p_l,\sigma_\lambda x))2u\phi_{i,\varepsilon}(x)\,dx\\
&=2\xi_{\sigma,i}^2\nabla_{\tau(p_i)}\nabla_{\tau(p_l)}G_{\sqrt{D}}(\sigma_\lambda p_l,\sigma_\lambda p_i)y_1\int_{\mathbb{R}^2_+}w\frac{\partial w}{\partial x_1}x_1\,dx+O(\sigma^3\xi_\sigma^2\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma}y^2)\\
&=2\xi_{\sigma,i}^2\nabla_{\tau(p_i)}\nabla_{\tau(p_l)}G_{\sqrt{D}}(\sigma_\lambda p_l,\sigma_\lambda p_i)y_1\int_{\mathbb{R}^2_+}w\frac{\partial w}{\partial x_1}x_1\,dx[\delta_{l,i-1}+\delta_{l,i+1}]\\
&+O(\sigma^3\xi_\sigma^2\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma}y^2)+O(\xi_\sigma^2\sigma^2\log\varepsilon(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2y)\\
&=2\xi_{\sigma,i}^2\nabla_{\tau(p_i)}\nabla_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_l,\sigma p_i)y_1\int_{\mathbb{R}^2_+}w\frac{\partial w}{\partial x_1}x_1\,dx[\delta_{l,i-1}+\delta_{l,i+1}]\\
&+O(\sigma^3\xi_\sigma^2\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma}y^2)+O(\xi_\sigma^2\sigma^2\log\varepsilon(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2y)+o(\xi_\sigma^2\lambda_\varepsilon y)
\end{aligned}$$

for $i\neq l$ and

$$\begin{aligned}
&(\psi_{l,\varepsilon}-\frac{\partial v}{\partial\tau(p_l)})(p_l+(y_1,0))-(\psi_{l,\varepsilon}-\frac{\partial v}{\partial\tau(p_l)})(p_l) \qquad (9.28)\\
&=\sum_{|i-l|=1}\xi_{\sigma,i}^2\nabla_{\tau(p_i)}\nabla_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_i,\sigma p_l)y_1\int_{\mathbb{R}^2_+}w^2\,dx\\
&+O(\sigma^3\xi_\sigma^2\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma}y^2)+O(\xi_\sigma^2\sigma^2\log\varepsilon(\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma})^2y)+o(\xi_\sigma^2\lambda_\varepsilon y).
\end{aligned}$$

Thus we have for $J_{4,l}$,

$$\begin{aligned}
J_{4,l}&=\frac{1}{\xi_\sigma}\sum_{i,j=1}^k\int_{\Omega_\varepsilon}a_{i,\varepsilon}\frac{u_j^2}{v^2}(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon})\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&=\frac{1}{\xi_\sigma}\sum_{i,j=1}^k\int_{\Omega_\varepsilon}a_{i,\varepsilon}\frac{u_j^2}{v^2}(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon})(p_l)\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&+\frac{1}{\xi_\sigma}\sum_{i,j=1}^k\int_{\Omega_\varepsilon}a_{i,\varepsilon}\frac{u_j^2}{v^2}[(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon})(x)-(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon})(p_l)]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&=J_{6,l}+J_{7,l}.
\end{aligned}$$

For $J_{6,l}$, we have from (9.26)

$$\begin{aligned}
J_{6,l} &= \frac{1}{\xi_\sigma}\sum_{i,j=1}^{k}\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_j^2}{v^2}\Big(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon}\Big)(p_l)\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&= \sum_{i}^{k}\frac{1}{\xi_\sigma}\Big(\frac{\partial v}{\partial\tau(p_l)}\delta_{il}-\psi_{i,\varepsilon}\Big)(p_l)\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_l^2}{v^2}\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&+\sum_{i=1}^{k}\sum_{j\neq l}\frac{1}{\xi_\sigma}\Big(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon}\Big)(p_l)\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_j^2}{v^2}\frac{\partial u_l}{\partial\tau(p_l)}\,dx = o(\xi_\sigma^2\sigma^2\frac{\epsilon D}{\xi_\sigma}).
\end{aligned}$$

Similarly, using (9.27), (9.28), (5.15) and (5.16), we get

$$\begin{aligned}
J_{7,l} &= \frac{1}{\xi_\sigma}\sum_{i,j=1}^{k}\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_j^2}{v^2}\Big[\Big(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon}\Big)(x)-\Big(\frac{\partial v}{\partial\tau(p_j)}\delta_{ij}-\psi_{i,\varepsilon}\Big)(p_l)\Big]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&= \frac{1}{\xi_\sigma}\sum_{i=1}^{k} a_{i,\varepsilon}\int_{\Omega_\varepsilon}\frac{u_i^2}{v^2}\Big[\Big(\frac{\partial v}{\partial\tau(p_i)}-\psi_{i,\varepsilon}\Big)(x)-\Big(\frac{\partial v}{\partial\tau(p_i)}-\psi_{i,\varepsilon}\Big)(p_l)\Big]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&-\sum_{i=1}^{k}\sum_{j\neq i}\frac{1}{\xi_\sigma}\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_j^2}{v^2}[\psi_{i,\varepsilon}(x)-\psi_{i,\varepsilon}(p_l)]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&= \frac{1}{\xi_\sigma}\int_{\Omega_\varepsilon} a_{l,\varepsilon}\frac{u_l^2}{v^2}\Big[\Big(\frac{\partial v}{\partial\tau(p_l)}-\psi_{l,\varepsilon}\Big)(x)-\Big(\frac{\partial v}{\partial\tau(p_l)}-\psi_{l,\varepsilon}\Big)(p_l)\Big]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&-\frac{1}{\xi_\sigma}\sum_{|i-l|=1}\int_{\Omega_\varepsilon} a_{i,\varepsilon}\frac{u_l^2}{v^2}[\psi_{i,\varepsilon}(x)-\psi_{i,\varepsilon}(p_l)]\frac{\partial u_l}{\partial\tau(p_l)}\,dx\\
&+o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma})\\
&= \Big(\int_{\mathbb{R}^2_+} w^2\,dy\int_{\mathbb{R}^2_+} w^2\frac{\partial w}{\partial y_1}y_1\,dy\Big)\Big[\sum_{|i-l|=1}\xi_{\sigma,i}^2\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_i,\sigma p_l)\Big]a_{l,\varepsilon}\\
&+\Big(2\int_{\mathbb{R}^2_+} w\frac{\partial w}{\partial y_1}y_1\,dy\int_{\mathbb{R}^2_+} w^2\frac{\partial w}{\partial y_1}y_1\,dy\Big)\\
&\times[\xi_{\sigma,l-1}^2a_{l-1,\varepsilon}\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_l,\sigma p_{l-1})+\xi_{\sigma,l+1}^2a_{l+1,\varepsilon}\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_l,\sigma p_{l+1})]\\
&+o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma})\\
&= \Big(\int_{\mathbb{R}^2_+} w^2\,dy\int_{\mathbb{R}^2_+} w^2\frac{\partial w}{\partial y_1}y_1\,dy\Big)\xi_\sigma^2(1+o(1))\times\Big[\Big[\sum_{|i-l|=1}\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_i,\sigma p_l)\Big]a_{l,\varepsilon}\\
&-[a_{l-1,\varepsilon}\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_l,\sigma p_{l-1})+a_{l+1,\varepsilon}\nabla^2_{\tau(p_l)}G_{\sqrt{D}}(\sigma p_l,\sigma p_{l+1})]\Big]+o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma}),
\end{aligned}$$

where we have used the relation

$$2\int_{\mathbb{R}^2_+} w\frac{\partial w}{\partial y_1}y_1\,dy = -\int_{\mathbb{R}^2_+} w^2\,dy.$$

Combining the above two estimates, we have

$$J_{4,l} = (\int\limits_{\mathbb{R}^2_+} w^2\,dy \int\limits_{\mathbb{R}^2_+} w^2 \frac{\partial w}{\partial y_1} y_1\,dy)\xi_\sigma^2(1+o(1)) \times \Big[[\sum_{|i-l|=1} \nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_i, \sigma p_l)]a_{l,\varepsilon}$$
$$-[a_{l-1,\varepsilon}\nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_{l-1}) + a_{l+1,\varepsilon}\nabla^2_{\tau(p_l)} G_{\sqrt{D}}(\sigma p_l, \sigma p_{l+1})]\Big] + o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma}),$$

and this proves (9.22).

Using

$$\frac{1}{R}\frac{\partial w}{\partial \alpha} = w'(|y|)(1+O(\epsilon\alpha))$$

and

$$\frac{1}{R^4} g^{(4)}(0) = \frac{\partial^2}{\partial \tau(\varepsilon p_i)^2} h(\varepsilon p_i)\,(1+O(\epsilon|p_i|)),$$

we can evaluate the integral $J_{5,l}$.

A computation analogous to (5.6) gives

$$\begin{aligned}
J_{5,l} &= \sum_{i=1}^k 2\varepsilon^3 a_{\varepsilon,i} \int\limits_{\Omega_\epsilon} \bar{w}_i \bar{v}^{(1)}_{R,i} \frac{\partial u_l}{\partial \tau(p_l)} \\
&= \xi_{\sigma,l}\varepsilon^3 a_{\epsilon,l} \int\limits_{B_R} 2w\bar{v}^{(1)}_R \frac{1}{r}\frac{\partial w}{\partial \alpha} r\,(1+O(\epsilon\alpha))\,dr\,d\alpha \\
&= \xi_{\sigma,l}\varepsilon^3 a_{\epsilon,l} \int\limits_{B_R} -\frac{1}{r}[(\Delta-1)\frac{\partial w}{\partial \alpha}]\bar{v}^{(1)}_{R,l}\, r\,(1+O(\epsilon\alpha))\,dr\,d\alpha \\
&= \xi_{\sigma,l}\varepsilon^3 a_{\epsilon,l} \int\limits_{\partial B_R} -\frac{1}{R}\left(\frac{\partial w}{\partial \alpha}\frac{\partial \bar{v}^{(1)}_{R,l}}{\partial b} - \bar{v}^{(1)}_{R,l}\frac{\partial}{\partial b}\frac{\partial w}{\partial \alpha}\right) R\,(1+O(\epsilon\alpha))\,d\alpha \\
&= \epsilon^3 \xi_{\sigma,l} a_{\epsilon,l} \int\limits_{\mathbb{R}} (w')^2 \frac{1}{6R^2} g^{(4)}(0)\frac{3y_1^2}{R^2}\,(1+O(\epsilon y_1))\,dy_1 \\
&= \epsilon^3 \xi_{\sigma,l} a_{\epsilon,l} \frac{3}{2R^4}\nu_1 g^{(4)}(0) + o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma}) \\
&= \epsilon^3 \xi_{\sigma,l} a_{\epsilon,l} \frac{3}{2}\nu_1 \frac{\partial^2}{\partial \tau(\varepsilon p_l)^2} h(\varepsilon p_l) + o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma}),
\end{aligned} \tag{9.29}$$

where the constant $\nu_1 > 0$ has been defined in (5.7). Thus

$$J_{5,l} = \epsilon^3 \xi_{\sigma,l} a_{l,\epsilon} \frac{3}{2}\nu_1 \frac{\partial^2}{\partial \tau(\varepsilon p_l)^2} h(\varepsilon p_l) + o(\xi_\sigma^2\sigma^2\frac{\varepsilon D}{\xi_\sigma}).$$

From the estimates on I_3 and I_4, we know that

$$\|\phi_\varepsilon^\perp\|_{H^2(\Omega_\varepsilon)} \le C\|I_3\|_{L^2(\Omega_\varepsilon)} \le C\xi_\sigma\sigma\frac{\varepsilon D}{\xi_\sigma}\log\frac{\varepsilon D}{\xi_\sigma}. \tag{9.30}$$

Next by the definition of $J_{1,l}$, using (9.30)

$$\begin{aligned} J_{1,l} &= \int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \frac{\partial v}{\partial \tau(p_l)} \phi_\varepsilon^\perp \, dx \\ &= O(\xi_\sigma^2 \sigma \frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma}) \|\phi_\varepsilon^\perp\|_{H^2(\Omega_\varepsilon)} \\ &= o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}). \end{aligned} \tag{9.31}$$

We have obtained (9.19). Next we estimate $J_{2,l}$:

$$\begin{aligned} J_{2,l} &= -\int_{\Omega_\varepsilon} \frac{u^2}{v^2} \psi_\varepsilon^\perp \frac{\partial u_l}{\partial \tau(p_l)} \, dx \\ &= -\int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \psi_\varepsilon^\perp \frac{\partial u_l}{\partial \tau(p_l)} \, dx + O(\varepsilon^4) \\ &= -\int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \psi_\varepsilon^\perp(p_l) \frac{\partial u_l}{\partial \tau(p_l)} \, dx - \int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \frac{\partial u_l}{\partial \tau(p_l)} (\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(p_l)) \, dx + O(\varepsilon^4) \\ &= -J_{8,l} - J_{9,l} + O(\varepsilon^4). \end{aligned}$$

By the equation satisfied by $\psi_\varepsilon^\perp$, we have

$$\begin{aligned} \psi_\varepsilon^\perp(p_l) &= \int_{\Omega_\varepsilon} G_{\sqrt{D}}(\sigma_\lambda p_l, \sigma_\lambda x) 2u \phi_\varepsilon^\perp(x) \, dx \\ &= O(\xi_\sigma \sigma \frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma}). \end{aligned}$$

Further,

$$\begin{aligned} \psi_\varepsilon^\perp(p_l + y) - \psi_\varepsilon^\perp(p_l) &= \int_{\Omega_\varepsilon} [G_{\sqrt{D}}(\sigma_\lambda(p_l + y), \sigma_\lambda x) - G_{\sqrt{D}}(\sigma_\lambda p_l, \sigma_\lambda x)] 2u \phi_\varepsilon^\perp(x) \, dx \\ &= \int_{\Omega_\varepsilon} [G_{\sqrt{D}}(\sigma_\lambda(p_l + y), \sigma_\lambda x) - G_{\sqrt{D}}(\sigma_\lambda p_l, \sigma_\lambda x)] 2 \sum_{j=1}^k u_j \phi_\varepsilon^\perp(x) \, dx + O(\varepsilon^4) \\ &= O(\xi_\sigma \sigma^3 \frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma} y) \end{aligned}$$

We have by the estimate for $\frac{\partial v}{\partial \tau(p_l)}$,

$$\begin{aligned} J_{8,l} &= \psi_\varepsilon^\perp(p_l) \int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \frac{\partial u_l}{\partial \tau(p_l)} \, dx \\ &= \psi_\varepsilon^\perp(p_l) (\int_{\Omega_\varepsilon} \frac{2}{3} \frac{u_l^3}{v^3} \frac{\partial v}{\partial \tau(p_l)} + O(\varepsilon^4)) \\ &= O(\xi_\sigma^3 \sigma^2 (\frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma})^2) = o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \end{aligned}$$

and

$$
\begin{aligned}
J_{9,l} &= \int_{\Omega_\varepsilon} \frac{u_l^2}{v^2} \frac{\partial u_l}{\partial \tau(p_l)} (\psi_\varepsilon^\perp(x) - \psi_\varepsilon^\perp(p_l))\, dx \\
&= O(\xi_\sigma^2 \sigma^3 \frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma}) = o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma})
\end{aligned}
$$

Thus we have

$$
J_{2,l} = o(\xi_\sigma^2 \sigma^2 \frac{\varepsilon D}{\xi_\sigma}), \tag{9.32}
$$

and (9.20) follows. Last we consider $J_{3,l}$,

$$
\begin{aligned}
J_{3,l} &= \lambda_\varepsilon \int_{\Omega_\varepsilon} \phi_\varepsilon^\perp \frac{\partial u_l}{\partial \tau(p_l)}\, dx \\
&= O(\xi_\sigma^2 \sigma \frac{\varepsilon D}{\xi_\sigma} \log \frac{\varepsilon D}{\xi_\sigma} \lambda_\varepsilon) = o(\xi_\sigma \lambda_\varepsilon).
\end{aligned} \tag{9.33}
$$

Thus (9.21) follows. □

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